# SINGULARITIES of SCHUBERT VAR'S WITHIN A RIGHT CELL

(j/w. P. McNamara)

<u>Main result</u>: Let $y, w \in S_n$ be such that

- $X_w \subset \mathcal{F}l_n$ Schubert var. is singular at $y$

Then: $\exists\, N \geq n$ $\exists\, \bar{y}, \bar{w} \in S_N$ s.t.

(i) $\mathcal{F}l_N \supset X_{\bar{w}}$ singular at $\bar{y}$ same sing. as above

(ii) $\bar{y} \underset{R}{\sim} \bar{w}$ (belong to same KL-right cell)

<u>Why should one care</u>?

ⓐ Reducibility of associated var's of hw modules (for $\mathfrak{sl}_n$)
( Qn Borho-Brylinski, Joseph ~85 )

ⓑ 0-1 Conjecture for KL-polynomials (type A)
( Qn/Obs Lascoux-Schützenberger '81 )
↪ counter-expl's: '03 McLarnan-Warrington

ⓒ Comparison of bases for irreps for $\mathcal{H}(S_n)$ ← Hecke alg.
(related p-KL-bases)

(2) $G \supset B \supset T \rightsquigarrow W = \text{Weyl gp}$

ss. alg gp/$\mathbb{C}$ — Borel — max'l torus

$X = G/B = \bigsqcup_{w \in W} \overbrace{BwB/B}^{C_w \text{ Schubert cell}}$

$\overline{C_w} =: X_w$ Schubert variety

$G = SL_n \rightsquigarrow X = \mathcal{F}l_n$

$\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$

$\mathfrak{h}^* \supseteq R = \underbrace{R^+}_{\text{pos. roots}} \sqcup R^- \qquad \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in \mathfrak{h}^*$

$\forall w \in W$

$\rightsquigarrow \underset{\substack{\| \\ L_w}}{L(-w\rho - \rho)}$ simple hw module

$V(L_w)$ "associated var." $\subseteq \mathfrak{g}^*$ (ir)reducible ?

( $L_w$ admits a filtration which is compatible with the PBW filtration of $\mathcal{U}(\mathfrak{g})$.

$\mathrm{gr}\, L_w$ is $\underset{\substack{\cong \\ \mathbb{C}[\mathfrak{g}^*]}}{S(\mathfrak{g})}$-fin. gen'd mod $\rightsquigarrow$ coh. sheaf on $\mathfrak{g}^*$

$V(L_w) =$ supp of such a sheaf )

$V(L_w) \subseteq \mathfrak{g}^*$

$\uparrow \mu$

$Ch(L_w) \subseteq T^*X \qquad \mu(Ch(L_w)) = V(L_w)$

$Ch(L_w) \subseteq \bigcup_{y \in W} \overline{T^*_{X_y} X}$

taking multiplicities $\rightsquigarrow$ $CC(L_w) = \sum \overbrace{m_{y,w}}^{\mathbb{Z}_{\geq 0} \ni} [\overline{T^*_{X_y} X}]$

$\boxed{m_{y,w} = ?}$ HARD!

- $m_{ww} = 1$
- $m_{yw} = 0$ unless $C_y \subseteq X_w$, $X_w$ is singular at $y$
- singularity invt.

Rmk $Ch(L_w)$ irr $\iff$ $m_{y,w} = 0 \ \forall y \neq w$

Conj. ① (type A) $Ch(L_w)$ is irred [Kazhdan-Lusztig '80]

False: [For other types, known since the beginning]

- 1997 Kashiwara-Saito: counterexample $y \in X_w \subseteq SL_8/B$
- Braden '01
- Vilonen-Williamson ('12) $m_{y,w} \leftrightarrow$ decomp. numbers for perverse sheaves

__Weaker Conj.__: (Beilinson–Brylinski, Joseph) Is $V(L_w)$ irred? (type A)

$$V(L_w) \text{ irred} \iff \boxed{m_{y,w} = 0 \ \forall\, y \underset{LR}{\sim} w \quad y \neq w}$$

⚠ From known examples of reducible $Ch(L_w)$
+ Main result
$\implies$ counterexamples to weaker conj.

2014: Williamson showed that the conj. is false

## Williamson's strategy

- Pick KS counterexample to Conj ①.
  This comes from a Schubert $X_w \subset SL_8/B$
  by looking at its singularity at $y \in X_w$
- Want to find same singularity in a Schubert variety $X_{\bar{w}}$ at $\bar{y}$ in such a way that $\bar{y} \underset{LR}{\sim} \bar{w}$

$m_{y,w}$ is uniquely determined by

$$B_- yB/B \cap X_w =: N_{y,w}$$ normal slice to $X_w$ along the Schubert cell

Look for $\bar{y}, \bar{w}$ s.t. $\quad \cong$

$$\tilde{B}_- \bar{y} \tilde{B}/\tilde{B} \cap X_{\bar{w}} =: N_{\bar{y},\bar{w}}$$

To find such a pair:

- use properties of $p$-canonical basis
- result of decomposition numbers for perverse sheaves (by Vilonen-Williamson)
- computer calculations (MAGMA: Howlett-Nguyen) to find potential counterexamples.

**Today: ELEMENTARY STRATEGY TO FIND $\underline{y}, \underline{w}$ WITH DESIRED PROPERTIES!**

## KL-Cells in $S_n$

Recall: The Robinson-Schensted correspondence:

$$S_n \longrightarrow SYT(n) \times SYT(n)$$

$$(x_1 \dots\dots x_n) = x \longmapsto (P(x), \underset{P(x^{-1})}{\underset{\shortparallel}{Q(x)}})$$

Defn (Thm KL'79, Garsia-McLarnan '88) Let $y, w \in S_n$

- $y \underset{R}{\sim} w$ if $P(y) = P(w)$
- $y \underset{RL}{\sim} w$ if $P(y)$ has same shape as $P(w)$

Rmk $y \underset{R}{\sim} w \Rightarrow y \underset{RL}{\sim} w$

Algorithm: INPUT: $y, w \in S_n$
OUTPUT: $\bar{y}, \bar{w} \in S_N$ where $\bar{w} \underset{R}{\sim} \bar{y}$

If $P(y) = P(w) \longrightarrow \bar{y} = y \quad \bar{w} = w$

Otherwise, $\exists\, k =$ min'l entry which lies in diff'nt boxes

$$\ell := \max\{c_y(k),\ c_w(k)\}$$

column index of box containing $k$ in $P(y)$

$x \in \{y, w\}$

$$x' = (k, k+1, \dots, k+\ell-2, x'(\ell), x'(\ell+1), \dots, x'(n+\ell-1))$$

$$j = 1, \dots, n \qquad x'(j+\ell-1) = \begin{cases} x(j) & \text{if } x(j) < k \\ x(j)+\ell-1 & \text{if } x(j) \geq k \end{cases}$$

Thm 1) The above algorithm terminates (after at most $n$ steps)
2) Let $\bar{y}, \bar{w}$ be the final output
then $m_{\bar{y}, \bar{w}} = m_{y, w}$

Cor From known reducible char. var's $\Longrightarrow$ resol. associated var's

Exercise: KS: $y = (2\,1\,6\,5\,4\,3\,8\,7) \rightsquigarrow (\bar{y}, \bar{w})$
$w = (6\,2\,8\,4\,5\,1\,7\,3)$

**Exemple.** $y = (2\,1\,4\,3)$ $\quad$ $w = (4\,2\,3\,1)$

$\emptyset \leftarrow 2 \leftarrow 1 \leftarrow 4 \leftarrow 3$ $\quad$ $\emptyset \leftarrow 4 \leftarrow 2 \leftarrow 3 \leftarrow 1$

$\boxed{2}\quad \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1 & 4\\ \hline 2 \\ \cline{1-1}\end{array}$ $\qquad$ $\boxed{4}\quad \begin{array}{|c|}\hline 2\\ \hline 4\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 2 & 3\\ \hline 4 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$

$P(y) = \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & 4\\ \hline\end{array}$ $\qquad$ $P(w) = \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$ $\qquad$ $k = 4$

$\ell = \boxed{c_y(4) = 2}$ $\qquad$ $c_w(4) = 1$

$y' = (4, 2, 1, 5, 3)$ $\qquad$ $w' = (4, 5, 2, 3, 1)$

$\boxed{4}\quad \begin{array}{|c|}\hline 2\\ \hline 4\\ \hline\end{array}\quad \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 4\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1 & 5\\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$ $\qquad$ $\boxed{4}\quad \begin{array}{|c|c|}\hline 4 & 5\\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 2 & 5\\ \hline 4 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 3\\ \hline 4 & 5\\ \hline\end{array}$

$$\begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & 5\\ \hline 4 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & 5\\ \hline 4 \\ \cline{1-1}\end{array}$$

$$N_{y,w} \cong \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ c & d & 1 & 0 \end{pmatrix} \,\middle|\, ad - bc = 0 \right\}$$

$$N_{\bar{y},\bar{w}} \cong \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & a & b & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ e & c & d & 1 & 0 \end{pmatrix} \,\middle|\, \begin{array}{l} e = 0 \\ ad - bc = 0 \end{array} \right\}$$