Weak generalized lifting property, Bruhat intervals and Coxeter matroids

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Coxeter matroids

- Generalization of Whitney’s (ordinary) matroids
- Introduced by I. Gelfand and V. Serganova in 1987
- Studied by many people such as V. Borovik, I. Gelfand, M. Goresky, R. MacPherson, V. Serganova, A. Vince, N. White, A. Zelevinsky...
- Lies at the intersection of Combinatorics, Algebra, Geometry, Optimization Theory

We want to tell you about:
**Theorem (Caselli-D’Adderio-M).** Bruhat intervals of finite Coxeter groups are Coxeter matroids

Main new tool in the proof of the theorem:
**Weak generalized lifting property**
(true for all finite and infinite Coxeter groups)
(W, S)  Coxeter system

- W Coxeter group
- S = \{s_1, \ldots, s_n\} Coxeter generators
- relations:
  \[ s_i^2 = e \quad \text{(involutions)} \]
  \[ (s_is_j)^{m_{ij}} = e \quad m_{ij} \in \mathbb{N}_{\geq 2} \cup \{\infty\} \]

Finite Coxeter groups are Reflection Groups
W acting on a real vector space V

\[ \ell(w) := \min\{k : w \text{ is a product of } k \text{ generators}\} \quad \text{length} \]

\[ \Phi = \Phi^+ \sqcup \Phi^- \quad \text{(positive and negative) roots} \]

T  reflections

\[ T \overset{\sim}{\longleftrightarrow} \Phi^+ \quad \text{bijection} \]
\[ t \quad \mapsto \quad \alpha_t \]
Bruhat order on $W$:

It is the transitive closure of $u < v$ if and only if there exists $t \in T$ such that:

$$\begin{cases} v = tu \\
\ell(v) = \ell(u) + 1 \end{cases}$$

Properties:

- The identity $e$ is the bottom element.
- The poset is ranked by length function $\ell$.
- There exists a top element iff $W$ is finite.

Hasse diagram: upword edge from $u$ to $v$ iff $u < v$.

We also label the edge with the positive root $\alpha_t$ corresponding to $t$. 

\[ u \rightarrow v = tu \]

\[ \alpha_t \]
EXAMPLE: $S_3$

$$(W, S = \{s_1, s_2\}) \quad \text{relations: } s_1^2 = s_2^2 = (s_1s_2)^3 = e$$

$W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 = s_2s_1s_2\} \simeq \text{symmetric group } S_3$$
$W$ of type $A_3$: the symmetric group $S_4$
Let \((W, S)\) be finite, \(J \subseteq S\).

- \(W_J := \langle J \rangle\)  parabolic subgroup generated by \(J\)
- \(W^J := \{w \in W : ws > w \ \forall s \in J\}\)  minimal left coset representatives

There is a unique decomposition \(W \cong W_J \times W_J\)

\[\begin{array}{c}
w \\
\downarrow \\
{\overset{\sim}{\longmapsto}} \\
{\overset{\sim}{\longmapsto}} \\
\downarrow \\
W^J \cong W/W_J \text{ bijection}
\end{array}\]

Fix \(p \in V\) s. t. \((p, \alpha_s)\)

\[\begin{cases}
eq 0 & \text{if } s \in J \\
< 0 & \text{if } s \notin J
\end{cases}\]

\[\delta_p : W/W_J \to V\]

\[vW_J \mapsto v(p)\]  well-defined since \(W_J\) fixes \(p\)

Given \(\emptyset \neq \mathcal{M} \subseteq W/W_J\), define a polytope associated with \(\mathcal{M}\):

\[\Delta_{\mathcal{M}}(p) = \text{convex hull of } \delta_p(\mathcal{M}) \quad \text{shorthand: } \Delta_{\mathcal{M}}(p) = \Delta_{\mathcal{M}}\]
If $W = S_n$, $J = \emptyset$, $\mathcal{M} = W$, then $\Delta_\mathcal{M}$ is the classical permutohedron.
**w-Bruhat order on W:**

\[ u \leq^w v \iff w^{-1}u \leq w^{-1}v \]

Note: \( \leq^c = \leq \)

**w-Bruhat order on \( W/W_J \):**

**Theorem/Definition:** Every \( A \in W/W_J \) has a \( \min^w \) and a \( \max^w \) w.r.t. \( \leq^w \).

Let \( A, B \in W/W_J, w \in W \). TFAE:

\[ \begin{align*}
&\blacktriangleright A \leq^w B \\
&\blacktriangleright \min^w A \leq^w \min^w B \\
&\blacktriangleright \max^w A \leq^w \max^w B \\
&\blacktriangleright a \leq^w b \text{ for some } a \in A \text{ and } b \in B
\end{align*} \]

\( \emptyset \neq \mathcal{M} \subseteq W/W_J \) is a **Coxeter matroid for \( W \) and \( J \)** if it satisfies the **Maximality Property**

\[ \blackblacksquare \text{ for all } w \in W, \text{ there exists } A \in \mathcal{M} \text{ s. t. } B \leq^w A \text{ for all } B \in \mathcal{M} \]
EXAMPLE: ORDINARY MATROIDS

- \((W, S)\) of type \(A_{n-1}\)
- \(W \simeq S_n\) the symmetric group on \([n] := \{1, \ldots, n\}\)
- \(S = \{s_1, s_2, \ldots, s_{n-1}\}\) with \(s_i = (i, i + 1)\).

If \(J = S \setminus \{s_k\}\) then
- \(W_J \simeq S_k \times S_{n-k}\)
- every \(b \in W/W_J\) corresponds to a subset \(B\) of \([n]\) of cardinality \(k\)

\[
\frac{W}{W_J} \overset{\sim}{\longrightarrow} \binom{[n]}{k} \quad \text{bijection}
\]

With these choices:

\(\{\text{Coxeter matroids for } W \text{ and } J\} = \{\text{ordinary matroids on } [n] \text{ of rank } k\}\)

The \(B\)'s are the bases of the matroid
The theorem translates the definition of a Coxeter matroid into geometric terms.

**Theorem (Gelfand–Serganova).** Let $\emptyset \neq \mathcal{M} \subseteq W/W_J$. TFAE

- $\mathcal{M}$ is a Coxeter matroid
- for every edge of $\Delta_{\mathcal{M}}$, there exists a reflection of $W$ that flips that edge
- every edge of $\Delta_{\mathcal{M}}$ is parallel to a root in $\Phi$

- One of the most important tool of the theory
- Geometric interpretation of Coxeter matroids as polytopes with certain symmetry property
- Surprisingly simple (although cryptomorphic) definition of a Coxeter matroid
- This is why roots play a fundamental role.
$W = S_3$ \quad J = \emptyset$
\[ M = \{s_1, s_2, s_1s_2, s_2s_1\} \] is not a Coxeter matroid.
\[ M = \{s_1, s_1s_2, s_2s_1, s_1s_2s_1\} \] is a Coxeter matroid.
**Theorem (Caselli-D’Adderio-M).**

Let $(W, S)$ be a finite Coxeter system. For all $J \subseteq S$ and all $x, y \in W^J$ with $x \leq y$, the parabolic Bruhat interval

$$\{z \in W^J : x \leq z \leq y\}$$

is a Coxeter matroid.

- In 2015, Kodama and Williams prove the theorem for $W$ of type $A$ and $J = \emptyset$. 
Let $\mathcal{M}$ be a Bruhat interval:

$$\mathcal{M} = [x, y] = \{ z \in W : x \leq z \leq y \}$$

$\Delta_{\mathcal{M}}$ is the **Bruhat interval polytope** corresponding to $\mathcal{M}$

To prove the theorem, we

- translate the problem into geometric terms using Gelfand–Serganova Theorem
- need to prove that the edges of $\Delta_{\mathcal{M}}$ are parallel to roots
- study actually all faces of $\Delta_{\mathcal{M}}$
- use several algebro-combinatorial tools in the theory of Coxeter groups
- use a new tool: the Weak Generalized Lifing Property
**Classical Lifting Property (Verma).** Let $u, v \in W$ with $u < v$ and $s \in S$. If $u \prec su$ and $sv \prec v$, then $su \leq v$ and $u \leq sv$

**Pros.** Characterizes Bruhat order. Has many consequences: e.g. the interval is closed under multiplication by $s$

**Cons.** For some $u, v \in W$, there are no such $s \in S$. 
**Generalized Lifting Property** For all $u, v \in W$ with $u < v$, there exists $t \in T$ s.t. $u \triangleleft tu \leq v$ and $u \leq tv \triangleleft v$

**Pros.** Existence of such $t$. It holds for $W = S_n$ (Tsukerman–Williams ’15) and, more generally, for $W$ simply laced (Caselli–Sentinelli ’17)

**Cons.** It doesn’t hold for $W$ not simply laced (Caselli–Sentinelli ’17)
**Weak Generalized Lifting Property (C-D-M)** Given \(u, v \in W\) with \(u < v\), let \(R_v = \{\alpha_t \in \Phi^+ : u \leq tv < v\}\) and \(R_u = \{\beta_r \in \Phi^+ : u < rv \leq v\}\). Then \(\text{Cone}(R_v) \cap \text{Cone}(R_u) \neq \{0\}\).

**Cons.** It is “weak”

**Pros.** It holds for all (finite and infinite) Coxeter systems
**Lemma.** Let $F$ be a face of $\Delta_{[x,y]}(p)$. If $F$ contains $u(p)$ and $v(p)$ for some subinterval $[u,v]$, then there exists a complete chain $C$ from $u$ to $v$ such that $z(p) \in F$ for all $z \in C$.

- By induction on $\ell(v) - \ell(u)$.
- Let $f \in V^*$ be such that $f = c$ is the hyperplane containing $F$, and $f < c$ is the halfspace containing $\Delta_{[x,y]}(p) \setminus F$.
- By the Weak Generalized Lifting Property

\[
\sum_{i \in I} b_i \beta_{r_i} = \sum_{j \in J} a_j \alpha_{t_j} \neq 0
\]

with $u \triangleleft r_i u \leq v$, $u \leq t_j v \triangleleft v$, and $a_j, b_i > 0$
Recall
\[ r_i(u(p)) = u(p) + c_i \beta_{r_i} \]
\[ t_j(v(p)) = v(p) - d_j \alpha_{t_j} \]
with \( c_i, d_j > 0 \)

Since all points \( r_i(u(p)) \) and \( t_j(v(p)) \) belong to \( \Delta_{[x,y]}(p) \)
\[
\begin{align*}
f(\beta_{r_i}) &\leq 0 \quad f(\alpha_{t_j}) \geq 0 \\
\end{align*}
\]
Thus \( f(\sum b_i \beta_{r_i}) = f(\sum a_j \alpha_{t_j}) = 0 \), and
\[
\begin{align*}
f(\beta_{r_i}) &= f(\alpha_{t_j}) = 0
\end{align*}
\]
therefore, all points \( r_iu(p) \) and \( t_jv(p) \) lie in \( F \).

By the induction hypothesis, there is a complete chain \( C' \) from \( r_1u \) to \( v \) such that \( z(p) \in F \) for all \( z \in C' \). Take the chain \( C = C' \cup \{u\} \).
GRAZIE!