Applications of numerical linear algebra for the study of complex networks and systems

Digraphs, non negative matrices, eigenvalues

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Summary

- Neighbours, connectivity, condensation digraph, periodicity.
- Adjacency matrix. Irreducible matrices and their graph characterization.
- Powers of non negative adjacency matrices.
- Primitive matrices and their graph characterization.
- Convergent and semi convergent matrices.
- Perron and Perron-Frobenius theorems and the dominant eigenvalue and eigenvector of primitive and irreducible matrices.
- Stochastic matrices: consensus and average consensus.
- Examples.

Nodes, edges, endpoints, neighbors, degree, source, sink

A digraph of order n is an ordered pair $G = (V, E)$, where V is a set with n elements called nodes and $E \subseteq V \times V$ is a set of ordered pairs of nodes called edges. If (u, v) is an edge, then the nodes u and v are defined endpoints of the edge.
If $v \in V$ then:

- $N^{\text{in}}(v) = \{u \in V | (u, v) \in E\}$ is defined the set of in-neighbors of v and $d^{\text{in}}(v) = |N^{\text{in}}(v)|$ is defined the in-degree of v. If $d^{\text{in}}(v) = 0$, then v is defined source;
- $N^{\text{out}}(v) = \{u \in V | (v, u) \in E\}$ is defined the set of out-neighbors of v and $d^{\text{out}}(v) = |N^{\text{out}}(v)|$ is defined the out-degree of v. If $d^{\text{out}}(v) = 0$, then v is defined sink.

Notice that if $(v, v) \in E$ then v cannot be a source or a sink since $d^{\text{in}}(v) \neq 0 \neq d^{\text{out}}(v)$.

Connectivity

- An oriented path of a digraph is an ordered sequence of at least two nodes such that any ordered pair of consecutive nodes in the sequence is an edge of the digraph.
- An oriented path is simple if the nodes in the sequence are distinct except possibly for the first and the last node in the sequence. The length of a simple path is the number of its distinct nodes.
- If the first and the last node of a simple oriented path coincide, the path is defined cycle. The cycles of length 1 are called loops.
- A digraph is acyclic if it contains no cycles.

Strongly connected digraph

- G possesses a globally reachable node if one of its nodes can be reached from any other node by traversing an oriented path. Notice that the condition is empty for a graph having just one node.
- If $v, w \in V$ are two nodes of a digraph G then $v \equiv w$ if $v = w$ or there is an oriented path from v to w and an oriented path from w to v. This defines an equivalence relation on V whose classes are known as strongly connected components of G. If G has globally reachable nodes then they necessarily belong to the same strongly connected component.
- If V is the only equivalence class of the relation then G is defined strongly connected.
- Notice that if $|V| = 1$ then G is always strongly connected.

Condensation digraph

- The condensation digraph of a digraph G, denoted with $C(G)$, is the digraph whose nodes are the strongly connected components (classes of equivalence of the relation $\equiv$) of G. There is an edge between two different nodes (equivalence classes) u and v of $C(G)$ if and only if there is an edge in G between one of the nodes of u and one of the nodes of v. $C(G)$ does not have loops.
- C(G) is, by construction, acyclic. Moreover G contains a globally reachable node w if and only if $C(G)$ contains a globally reachable node (the class of equivalence of its globally reachable nodes).
- An acyclic digraph is the condensation graph of itself.

Acyclic digraphs

The following theorem holds in particular for condensation digraphs.

Theorem

An acyclic digraph G contains a globally reachable node if and only if it contains a unique sink.

Proof If G contains a globally reachable node this, due to acyclicity, has to be unique and has to be a sink. On the other hand, let G contain a unique sink, then this node has to be globally reachable. Actually, all the nodes from which the sink cannot be reached have out-degree different from zero (since they are not sinks) and some of them have to form a cycle and this is impossible in an acyclic digraph.

Periodicity

- If G is digraph with n nodes then it has only a finite number of cycles bounded by $\sum_{k=1}^{n}(k-1)!\binom{n}{k}$. The bound can be reached if $E = V \times V$.
- A digraph is aperiodic if there is no integer $p > 1$ that divides the length of every cycle of the digraph, otherwise periodic. As a trivial example, a digraph with a loop is aperiodic. The period of a digraph is defined as the maximum common divisor of the lengths of its cycles.
- An acyclic digraph is not aperiodic (the above condition becomes empty) and we can conventionally assume that its period is ∞.

Weighted digraphs

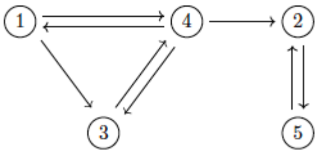
- If A is a square matrix of order n, let $D(A)$ the digraph such that $V = \{1, \ldots, n\}$ and $(i, j) \in E$ if $a_{i,j} \neq 0$. Sometimes it is useful associate $a_{i,j}$ with the edge (i, j). More formally the triple $G = (E, V, A)$ is defined weighted digraph, and $a_{i,j}$ weight of the edge (i, j). The matrix A is known as adjacency matrix of G.
- The two vectors Ae and $A^T e$ contain, respectively, the weighted out-degrees and in-degrees of the nodes of G.

Irreducible matrices

- If $G = D(A)$ is strongly connected the matrix A is defined irreducible and reducible otherwise. Notice that when $n = 1$ the graph G is always strongly connected so that the matrix A is always irreducible. Hence, when dealing with irreducible matrices, it is customary to assume that $n \geq 2$.
- A is irreducible if and only if A^T is irreducible.
- In the case where G is not strongly connected it can be shown that there exists a permutation matrix P such that PAP^T has a block upper triangular form with irreducible square diagonal blocks corresponding to the nodes that made up the strongly connected components of G.

Example

As an example consider the following graph G



G has two strongly connected components namely $\{1, 3, 4\}$ and $\{2, 5\}$.
Notice that $G = D(A)$ where

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow PAP^T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence A is reducible. It is worth to observe that, in addition, it is aperiodic.

Non negative matrices

Definition

A real matrix A is non negative (positive) if $a_{i,j} \geq 0$ ($a_{i,j} > 0$) for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. A real vector v is non-negative (positive) if $v_i \geq 0$ ($v_i > 0$) for $i = 1, \ldots, n$.

With $A \geq 0$ ($A > 0$), and $v \geq 0$ ($v > 0$) we denote respectively a non negative (positive) matrix A and vector v.

Powers of non negative adjacency matrices

If $G = D(A)$ and $A \geq 0$ the powers of A are strictly related to the oriented paths of G. The relation becomes stronger if the entries of A are in $\{0, 1\}$ (binary adjacency matrix).

Theorem

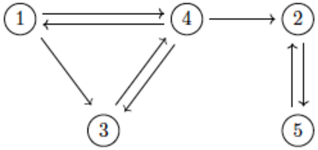
Let $G = D(A)$ and let $k \geq 1$:

- if $A \geq 0$ then the (i, j) entry of A^k is positive if and only if there is a path (not necessarily simple) in G made up of a sequence of $k + 1$ nodes, starting from i and ending with j;
- if the entries of A are in $\{0, 1\}$ then the (i, j) entry of A^k is the number of paths (not necessarily simple) in G of $k + 1$ nodes, starting from node i and ending with node j.

This theorem is certainly one of the reasons of the importance of non negative matrices.

Example

For the graph G of the previous example



one obtains

$$A^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad A^3 = \begin{bmatrix} 1 & 1 & 2 & {\color{red}2} & {\color{green}1} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

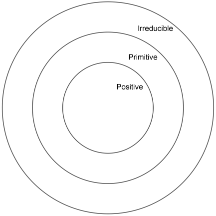
As an example, notice that the two paths from node 1 to node 4 made up by a sequence of four nodes (red entry in A^3) are not simple, while the analogous path from 1 to 5 (green entry) is simple.

Classes of non negative matrices

Definition

A matrix A is primitive if $A \geq 0$ and there exists $k \in \mathbb{N}$ such that $A^k > 0$.

If A is primitive certainly there exists a path (not necessarily simple) between two arbitrary chosen nodes i and j of $G = D(A)$ made up of a sequence of $k+1$ nodes, starting from i and ending with j. Hence G is strongly connected and A irreducible. On the other hand, a matrix can be irreducible without being primitive. A simple example is the matrix $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.



Graph characterization of primitive matrices

We know that a matrix A is irreducible if $G = D(A)$ is strongly connected. The following theorem furnishes a graph characterization for primitive matrices.

Theorem

Let $A \geq 0$ a matrix of order n and let $G = D(A)$. The matrix A is primitive if and only if G is strongly connected and aperiodic.

Notice that, when $n = 1$, G is strongly connected, and is aperiodic when $A \neq [0]$ (G is acyclic when $A = [0]$) so that the theorem holds. Hence, in the following we will consider the case where $n \geq 2$.

Frobenius number

In order to prove the previous result we need a lemma that recalls Bezout identity (the $\gcd$ of two integers is their integer linear combination). Given a set $S = \{a_1, \ldots, a_n\}$ of positive integers, an integer M is said to be representable by S if there exists non-negative integers $\{\alpha_1, \ldots, \alpha_n\}$ such that $M = \sum_{k=1}^{n} \alpha_k a_k$.

Lemma

The elements of S are coprime ($\gcd = 1$) if and only if there exists a largest unrepresentable integer, known as the Frobenius number of S.

If the elements of S have a nontrivial $\gcd$ then the Frobenius number cannot exist, so that the interesting implication is the other one.

Frobenius number

First of all let us prove the lemma in the case where $n = 2$. Let $S = \{a, b\}$ with $\gcd(a, b) = 1$. The Bezout identity states that there exist two integers $u > 0$ and $v \geq 0$ such that, say, $au - bv = 1$. Hence, for every integer $k \geq 1$ we have $aku - bkv = k$. In order to "control" the dependence form k of the coefficient of b, we perform a division with remainder and we obtain $kv = aq + r$, where $0 \leq r < a$. Hence $a(ku - bq) - br = k$ and $ku - bq > 0$, otherwise $a(ku - bq) - br \leq 0$ while $k > 0$. It follows that

$$a(ku - bq) + b(a - r) = ab + k.$$

Summarizing, since all the integers $> ab$ are representable, the Frobenius number of S is $\leq ab$ and in the linear combination the coefficient of b can be chosen $\leq a$.

Frobenius number

In the case where $n > 2$ it is possible to proceed by induction. To illustrate the idea, let us consider as an example the case where $S = \{6, 10, 15\}$. We want to find three nonnegative integers x, y, z such that $6x + 10y + 15z = n$, for every integer n above a certain bound. Since $6x + 10y = 2(3x + 5y)$ we can set $3x + 5y = w$. The inductive hypothesis tell us that we can find non-negative x and y in order to satisfy this equation if w is above a certain bound, say 15. Now, let us consider the equation $2w + 15z = n$. If n is above a certain bound, say 30, we can solve this equation and with $z \leq 2$. But if we choose $(n - 15 \cdot 2)/2 > 15$, i.e., $n > 60$, we are sure that exist x and y that give us the needed value of w.

Characterization of primitive matrices

Of course if $A \geq 0$ is primitive then every row of A has at least a positive entry. Hence, $A^k > 0$ implies $A^m > 0$ for every $m \geq k$. This implies that, for every m above a certain bound, there is a path with m successive edges from any node of $G = D(A)$ and itself. This is impossible if G is periodic, since these paths would be the combinations of cycles having $\mathrm{gcd} > 1$ and thus m should be a multiple of that gcd.

Characterization of primitive matrices

Now, let us assume that $G = D(A)$ is strongly connected and aperiodic. Given two nodes i and j of G there is an oriented path from i to j that contains a node of every cycle of G (remember that G has a finite number of cycles). Hence we can construct a new path from i to j by repeating each of the cycles arbitrarily. By using the lemma on Frobenius number we see than we can construct paths from i to j made up by a sequence of nodes above a certain bound $M(i, j)$. By taking the maximum over i and j of these bounds we obtain the thesis.

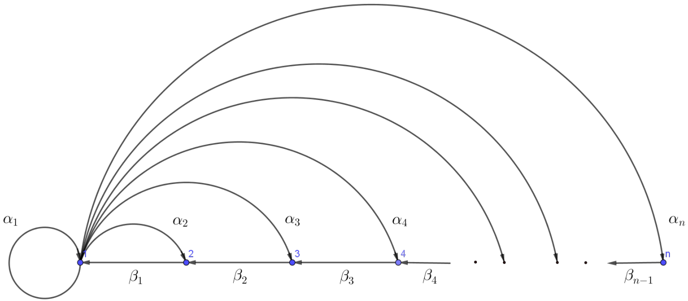
Example: Leslie population model

The Leslie population model is used to model the changes in a population of organisms over a period of time. The population is divided in $n \geq 2$ age classes indexed from $i = 1$ (the newborns) to $i = n$. The number of individuals in the i-th class at time k is denoted with $x_i(k)$. At every time step the $x_i(k)$ individuals (a) produce $\alpha_i x_i(k)$ offsprings, where $\alpha_i \geq 0$ is a fecundity rate and (b) progress to the next class with a survival rate $\beta_i \in [0, 1]$. The model can be described by the discrete time linear dynamical system $x(k) = A_L x(k-1)$ with

$$A_L = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ \beta_1 & 0 & \cdots & 0 & 0 \\ 0 & \beta_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \vdots & \beta_{n-1} & 0 \end{bmatrix}.$$

Example: Leslie population model

The graph $G = D(A_L)$ immediately suggests that A_L is irreducible if and only if $\alpha_n \neq 0$ and $\beta_i \neq 0$ for $i = 1, \dots, n-1$. The matrix A_L is primitive if and only if, besides being irreducible, there are $\alpha_i \neq 0$ and $\alpha_j \neq 0$ such that i and j are coprimes (this includes the case where $i = j = 1$).



Convergent and semi convergent matrices

Clearly, $x(k) = A_L x(k-1)$ implies $x(k) = A_L^k x(0)$. Hence, in order to obtain information on the evolution of $x(k)$, it is useful to study the behaviour of the powers of a matrix. Remember that a matrix A is defined:

- semi convergent if $\lim_{k \to +\infty} A^k$ exists and is a specific matrix;
- convergent if it is semi-convergent and the limit is the zero matrix;
- not convergent otherwise.

Eigenvalues, spectrum, spectral radius

In order to decide between these possibilities the set

$$\sigma(A) = \{\lambda | \lambda \text{ is an eigenvalue of } A\},$$

defined spectrum of A, is important. Frequently $\sigma(A)$ is not completely known but useful information can be obtained on

$$\rho(A) = \max\{|\lambda| | \lambda \in \sigma(A)\},$$

defined spectral radius of A.

Remember that an eigenvalue is

- semisimple if its algebraic and geometric multiplicities coincide,
- simple if it is semisimple and the two multiplicities are equal to 1.

Non negative matrix technology

By means of the Jordan Canonical Form (JCF) it can be shown that

- A is convergent if and only if $\rho(A) < 1$;
- A is semi convergent if and only if $\rho(A) = 1$, 1 is a semisimple eigenvalue of A and if $\lambda \in \sigma(A)$ and $\lambda \neq 1$ then $|\lambda| < 1$.

For non negative matrices important information on the spectral radius, can be obtained by means of Perron and Perron-Frobenius theorems. They are the cornerstones of what we can define as non negative matrix technology.

Perron theorem

For the sake of synthesis we state Perron theorem in a slightly generalized form.

Perron Theorem

1. If $A \geq 0$ then
   - there exists $\lambda \in \sigma(A)$ such that $\lambda = \rho(A) \geq 0$;
   - there exists a vector $0 \neq v \geq 0$ such that $Av = \lambda v$.
2. If A is primitive (in particular if $A > 0$), then
   - there exists $\lambda \in \sigma(A)$ such that $\lambda = \rho(A) > 0$;
   - λ is simple;
   - if $\mu \in \sigma(A)$ and $\mu \neq \lambda$ then $\lambda > |\mu|$;
   - there exists a vector $v > 0$ such that $Av = \lambda v$.

The first part of the theorem has the weakest possible hypothesis and a somewhat weak thesis. The second part has strong hypothesis and thesis.

Perron-Frobenius theorem for irreducible matrices

For irreducible matrices an intermediate result holds.

Perron-Frobenius Theorem

If $A \geq 0$ is an irreducible matrix of order $n \geq 2$ then

- there exists $\lambda \in \sigma(A)$ such that $\lambda = \rho(A) > 0$;
- λ is simple;
- there exists a vector $v > 0$ such that $Av = \lambda(A)v$.

Notice that the condition $n \geq 2$ is necessary since $A = [0]$ is irreducible having order 1. Actually an irreducible but non primitive matrix can have eigenvalues different from $\lambda = \rho(A)$ but with the same modulus. For example $\sigma(J) = \{1, -1\}$ so that $1 = \rho(J) = |-1|$.

Example

As an example of application of these theorems, let us prove a useful double inequality on the spectral radius of a non-negative matrix.

Theorem (double inequality)

If $A \geq 0$ then $\min Ae \leq \rho(A) \leq \max Ae$. Moreover if A is irreducible and $\min Ae < \max Ae$ then $\min Ae < \rho(A) < \max Ae$.

Proof First of all let us use Perron Theorem for non-negative matrices. As we know $\rho(A)$ is an eigenvalue of A and of A^T. Let $v \geq 0$ be an eigenvector of A^T associated to $\rho(A)$ and let us assume $v^T e = 1$. Hence

$$v^T A = \rho(A) v^T \Rightarrow v^T Ae = \rho(A).$$

Notice that $v^T Ae$ is a convex combination of the entries of Ae so that $\min Ae \leq \rho(A) \leq \max Ae$. If A is irreducible then $v > 0$. Let $(Ae)_i = \min Ae < \max Ae = (Ae)_j$. Since $v > 0$ it follows that $(Ae)_i < v^T Ae = \rho(A) < (Ae)_j$.

Example: Leslie population model

- If $\beta_i = 1$ for $i = 1, \ldots, n-1$ and $\sum_{k=1}^{n} \alpha_k > 1$ then $\rho(A_L) \geq 1$ and $\rho(A_L) > 1$ if A_L is irreducible (this means that A_L is not convergent and that it is possible to choose $x(0)$ in such a way that $\|x(k)\| \to +\infty$).
- If $\beta_i < 1$ for some i and $\sum_{k=1}^{n} \alpha_k \leq 1$ then the inequality $\rho(A_L) \leq \max A_L e$ implies $\rho(A_L) \leq 1$ and $\rho(A_L) < 1$ if A_L is irreducible (this means that A_L is convergent so that $\|x(k)\| \to 0$ for every choice of $x(0)$).

If $\beta_i = 1$ for $i = 1, \ldots, n-1$ and $\sum_{k=1}^{n} \alpha_k = 1$, the matrix A_L belongs to the important class of row stochastic matrices.

Stochastic matrices

Definitions

Let e be a vector whose n entries are all equal to one. A real square matrix $A \geq 0$ of order n is

1. row stochastic if $Ae = e$;
2. column stochastic if A^T is row stochastic;
3. doubly stochastic if row and column stochastic.

If $v \geq 0$ and $e^T v = \sum_i v_i = 1$, then v is defined probability vector since its entries are between 0 and 1 and can be interpreted as probabilities.

Spectral properties of stochastic matrices

If A is row (or column) stochastic:

- $1 \in \sigma(A) = \sigma(A^T)$;
- the double inequality theorem implies $\rho(A) = 1 = \rho(A^T)$;
- if A is primitive 1 is simple and $1 \neq \lambda \in \sigma(A)$ implies $|\lambda| < 1$.

Hence A cannot be convergent but can be semi convergent, and this happens in particular if A is primitive. For example

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is doubly stochastic and semi-convergent,

$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is doubly stochastic and not convergent.

$A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$ is primitive and thus semi-convergent.

Primitive stochastic matrices

If A is row stochastic and primitive, by means of JCF (or with a simpler proof if A is diagonalizable) we obtain an "explicit" formula for $\lim_k A^k$ by using the vector $w > 0$ is such that $w^T A = w^T$ normalized in such a way that $w^T e = 1$.

- If A is primitive and row stochastic then
$$\lim_{k\to+\infty} A^k = ew^T.$$
- If A is primitive and column stochastic then
$$\lim_{k\to+\infty} A^k = \lim_{k\to+\infty} ((A^T)^k)^T = we^T.$$
- If A is primitive and doubly stochastic then $w = e/n$ so that
$$\lim_{k\to+\infty} A^k = \frac{1}{n} ee^T.$$

Consensus and average consensus

If $x(k) = A^k x(0)$ then:

- if A is primitive and row stochastic

  $$\lim_{k\to+\infty} x(k) = ew^T x(0) = (w^T x(0))e,$$

  the system reach consensus (all the entries of x converge to the same convex combination of the entries of $x(0)$);
- if A is primitive and doubly stochastic

  $$\lim_{k\to+\infty} x(k) = \frac{1}{n} ee^T x_0 = \frac{e^T x(0)}{n} e,$$

  the system reach average consensus (all the entries of x converge to the arithmetic mean of the entries of $x(0)$).

Example: Leslie population model

If A_L is row stochastic and irreducible ($\alpha_n \neq 0$) then a simple direct computation shows that

$$w_i = \frac{\sum_{k=i}^{n} \alpha_k}{\sum_{k=1}^{n} k\alpha_k}.$$

This is the positive eigenvector whose existence is guaranteed by Perron-Frobenius theorem. Notice that

$$w_1 \geq w_2 \geq \ldots \geq w_n.$$

If A_L is primitive and we consider the system $x(k) = A_L x(k-1) = A_L^k x(0)$ then

$$\lim_{k \to +\infty} x(k) = (w^T x(0))e,$$

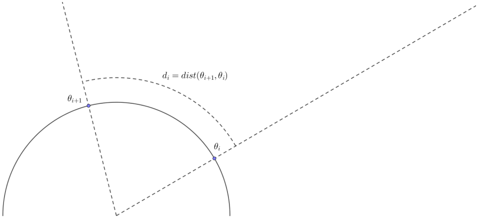
so that all the age classes converge towards the same number of individuals, and the first entries of $x(0)$ are the more important for the determination of this number.

Example: n-bugs system

A group of n robots, informally called “bugs”, are restricted to move on a circle of unit radius. The bugs are numbered with $i = 1, \ldots, n$ and their positions on the circle are individuated by angles $0 \leq \theta_i < 2\pi$ measured counterclockwise from the positive horizontal axis. In this setting it is easy to express the separation between two successive bugs as

$$d_i = \text{mod}(\theta_{i+1} - \theta_i, 2\pi), \qquad i = 1, \ldots, n$$

where it is understood that $n + 1$ has to be identified with 1.



Example: n-bugs system

We consider a situation where each bug feels an attraction towards its nearest counterclockwise neighbor proportional to their separation so that

$$\theta_i(k+1) = \mathrm{mod}(\theta_i(k) + \gamma d_i(k), 2\pi),$$

where $0 \leq \gamma \leq 1$ is a suitable constant. It is interesting to rewrite the dynamical system in terms only of pairwise separations. We have

$$\begin{aligned}
d_i(k+1) &= \mathrm{mod}(\theta_{i+1}(k+1) - \theta_i(k+1), 2\pi) \\
&= \mathrm{mod}(\theta_{i+1}(k) + \gamma d_{i+1}(k) - \theta_i(k) - \gamma d_i(k), 2\pi) \\
&= \mathrm{mod}((1-\gamma) d_i(k) + \gamma d_{i+1}(k), 2\pi) \\
&= (1-\gamma) d_i(k) + \gamma d_{i+1}(k),
\end{aligned}$$

where the last equality is justified since a convex combination of two numbers in $[0, 2\pi)$ results in a number in the same interval.

Example: n-bugs system

In matrix form we obtain the system $d(k) = Ad(k-1)$ where

$$A = \begin{bmatrix} 1-\gamma & \gamma & \cdots & 0 & 0 \\ 0 & 1-\gamma & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1-\gamma & \gamma \\ \gamma & 0 & \cdots & 0 & 1-\gamma \end{bmatrix}.$$

The matrix A is doubly-stochastic. If $\gamma = 0$ then $A = I$ and the bugs remain in their initial position. If $0 < \gamma < 1$ then it is not difficult to show that A is primitive in such a way that the system reach average consensus: the limiting separations between the bugs are all equal to the mean of the original separations. If $\gamma = 1$ then A is irreducible but not semi convergent.

Invariant probability vector

In the case where A is column stochastic the recurrence $x(k) = Ax(k-1)$ implies $e^T x(k) = e^T x(k-1)$. Hence if $x(0)$ is a probability vector then $x(k)$ is a probability vector for every k. This kind of recurrences are important in the study of Markov chains where the entries of $x(k)$ are the probabilities with which a random walker is in the various states after k transitions. Clearly if $x(0)$ is a probability vector then

$$\lim_{k \to +\infty} x(k) = we^T x(0) = w.$$

The vector w is known as invariant probability vector and its entries can be interpreted as the probabilities with which a random walker is in the various states in the long run. For this reason they are frequently exploited for measuring the importance, or centrality, of the states.

Conclusions

- There are important graph interpretations of certain matrix properties, in particular for non-negative matrices.
- In order to decide of the evolution of a system $x(k) = Ax(k-1)$ it is useful to understand the behaviour of the powers of A, that depends on its spectral properties.
- Perron and Perron-Frobenius theorems are fundamental tools for the study of spectral properties of primitive and irreducible matrices respectively.
- Stochastic matrices are a very important subclass of non-negative matrices.
  - If A is row stochastic the system $x(k) = Ax(k-1)$ can evolve to consensus.
  - If A is column stochastic and $x(0)$ is a probability vector the system $x(k) = Ax(k-1)$ can evolve to a vector known, in Markov chain theory, as invariant probability vector.

Suggested readings

- F. Bullo
  Lectures on network systems
  Create space, 2018.
- C. D. Meyer
  Matrix analysis and applied linear algebra
  SIAM, 2010.

Exercises

1. Explain the formula that expresses the bound on the number of cycles in a directed graph.
2. Prove that if A is reducible then all its powers are reducible.
3. Let $A \geq 0$ be irreducible. (a) If A is periodic can it have an aperiodic power? (b) If A is aperiodic can it have a periodic power?
4. Explain how induction can be applied in order to prove the lemma on Frobenius number if $S = \{a_1, \ldots, a_n\}$.
5. Why the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is primitive? Determine its first positive power.
6. Prove the formula for the vector w such that $w^T A_L = w^T$ if A_L is row stochastic and irreducible.
7. What happens in $\gamma = 1$ for the n-bugs system?