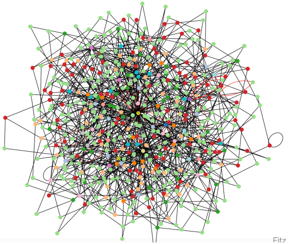
Evolution of distances in preferential attachment models

Júlia Komjáthy & Joost Jorritsma

A day on random graphs 30 June 2022



Properties of real contact networks

Most large real networks are *statistically very similar*. If you understand one, you understand others.

[Adapted from *Network Science* (2015) by Albert László Barabási]

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$$\log \mathbb{P}(\deg(v) = x) \asymp \log C - \tau \log x$$

log(proportion of degree x vertices) vs $\log x$ is a straight line.

Power laws

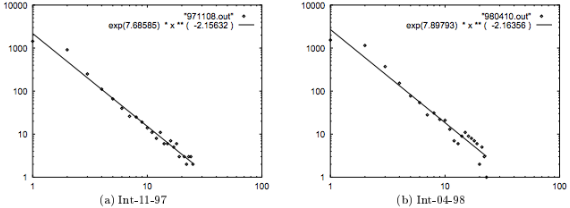
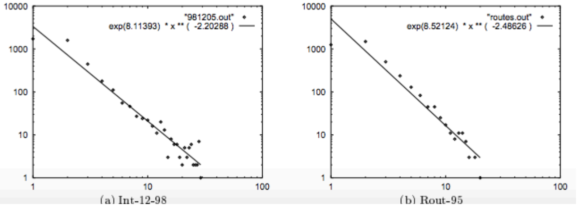


Figure 5: The outdegree plots: Log-log plot of frequency f_d versus the outdegree d.



Log-log plot of degree distribution of the router level internet network

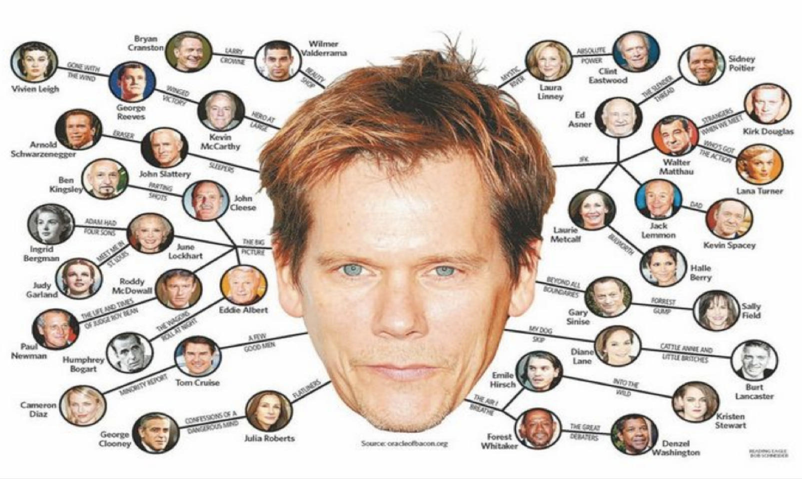
from Faloutsos, Faloutsos, Faloutsos. 1999

Growing networks

A question:

How did the network evolve around the servers of ‘99?

Small worlds and ultra-small worlds



Kevin Bacon game: Movie networks are small worlds

from Mark Robinson Writes

Small worlds and ultra-small worlds

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A network G on N vertices is a *small world* if the average distance

$$\overline{\mathsf{dist}}(G) = \frac{1}{\binom{N}{2}} \sum_{u,v \in G} d_G(u,v) = \Theta(\log N).$$

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Alternative def of ultrasmall:

$$\overline{\mathsf{dist}(G)} = \Theta(\log \log n)$$

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Static network models

Degree distribution is superimposed on the network

- Configuration model
- Inhomogeneous random graphs
- Chung-Lu, Norros-Reitu models

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Try to intrinsically explain a property of real life networks

- Original Barabási-Albert model aka Preferential attachment model
- Variable degree Preferential attachment model
- Bianconi-Barabási model aka PA with multiplicative fitness
- PA with additive fitness
- PA with power of choice
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- Heuristically:

$$\mathbf{Prob}(v_{N+1} \longrightarrow v_i) \propto \mathsf{deg}_N(v_i)$$

The original Preferential Attachment Model

Classic variations

- Fixed outdegree: $(1, 0)$ edge/new vertex:

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Animation

Degree distribution

Theorem (Degree sequence of PA(m,0))

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Limiting degree distribution of Classic PA $(m, 0)$

$$\lim_{N \to \infty} \mathbf{Prob}(\deg(V_N) = k) \asymp \frac{1}{k^3}.$$

where V_N *uniformly chosen vertex in* PA_N.

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Generally

Limiting degree distribution is

$$\lim_{N\to\infty} \mathbf{Prob}(\deg(V_N) = k) \asymp \frac{1}{k^\tau},$$

where

$$\tau_{m,\delta} = 3 + \delta/m \qquad \tau_f = 1 + 1/\gamma_f.$$

Graph distances

Theorem (Distances in PAMs, $\tau \in (2,3)$)

Dommers, v/d Hofstad, Hoogiemstra '10 & Dereich and Mörters '13

Let U_N, V_N be two uniformly chosen vertices (within the giant component)[1]*. When $\tau \in (2,3)$,*

$$d_G^{(N)}(U_N, V_N) = (1 + o_{\mathbf{P}}(1)) \log\log N \cdot \frac{4}{|\log(\tau - 2)|}$$

Lower tightness holds.

[1] $d_G^{(N)}$ graph distance within PA_N

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$\tau > 3$

Dommers, v/d Hofstad, Hoogiemstra '10

Typical distance in $\mathrm{PA}(m, \delta)$ for $m \geq 2$, $\tau > 3$:

$$\mathsf{d}_{\mathsf{G}}^{(\mathsf{N})}(U_N, V_N) = \Theta(\log N).$$

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Universality

For static models with power law degrees, $\tau \in (2, 3)$

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Factor 2:
In PA, a high degree vertex tends to be connected via a path of 2 to a higher degree vertex.
In static network models, directly.

Back to the question:

How do distances shrink as time passes?

Evolution of graph distances

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Jorritsma, Komjáthy, (AoAP, 22)

Let U_N, V_N be two uniformly chosen vertices in PA_N (within the giant component). When $\tau \in (2,3)$, & $m \ge 2$, for $t > N$:

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$$d_G^{(t)}(U_N, V_N) \asymp \Big(\log\log N - \log(\log(t/N) \vee 1)\Big) \cdot \frac{4}{\log(\tau-2)|} \vee 2$$

+*tightness around the main term.*

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$(X_N)_{N\geq 1}$ is a tight sequence of random variables.

Message 1: Fix $a \in (0,1)$. Then at $t = N \exp\left(\varepsilon(\log N)^a\right) \ll N^{1+\varepsilon}$,

$$d_G^{(t)}(U_N, V_N) \approx (1-a) \log\log N \cdot \frac{4}{|\log(\tau-2)|}.$$

Message 2: $d_G^{(t)}(U_N, V_N)$ never leaves a (tight) strip around the main term.

Proofs

Lower bound

Lower bound in growing network

- A path counting method:
  A **possible path** π is a sequence of labelled vertices. The path's (potential) arrival time is the youngest vertex on the path. Among all possible labeled paths π_{U_N,V_N},

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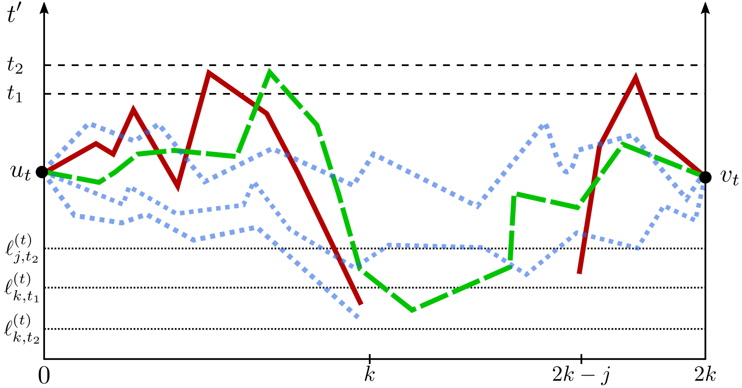
- A union bound not over time, but over possible paths.
- This still does not tend to zero...

Bad paths



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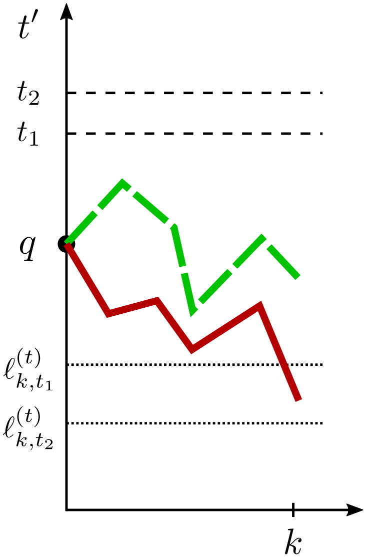

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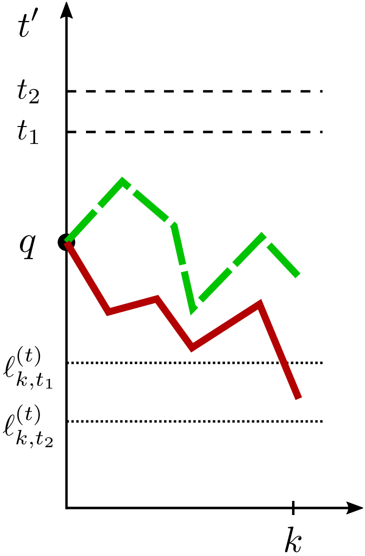

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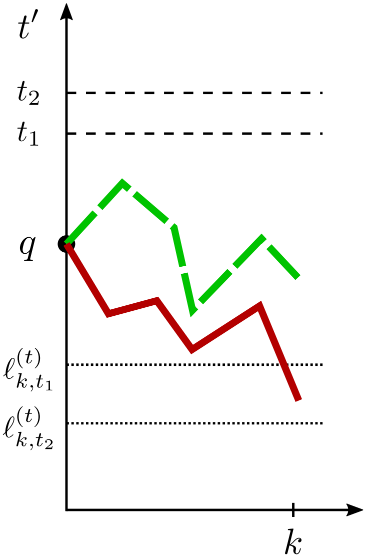

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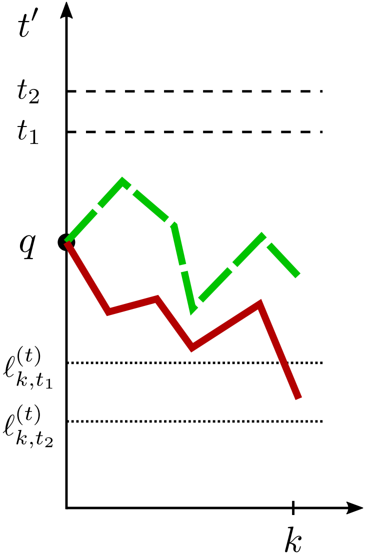

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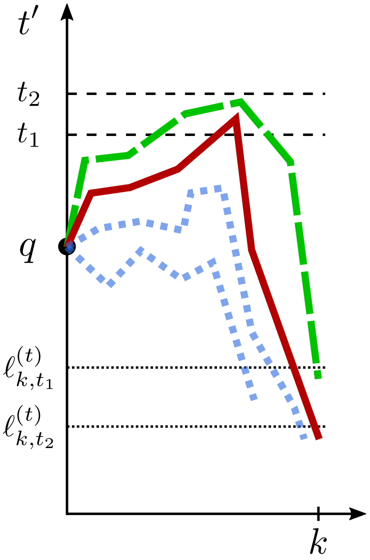

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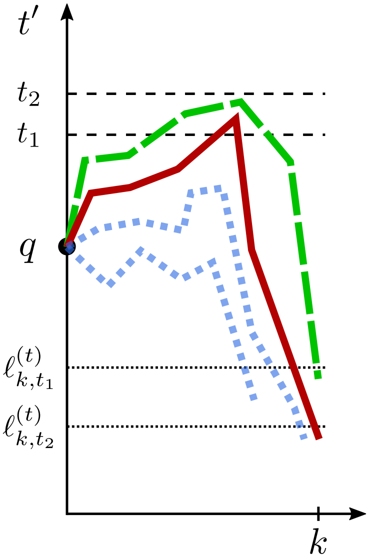

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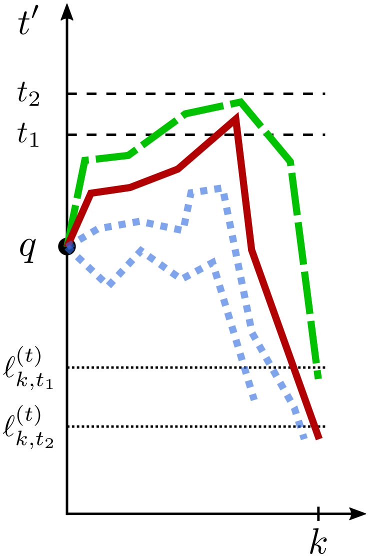

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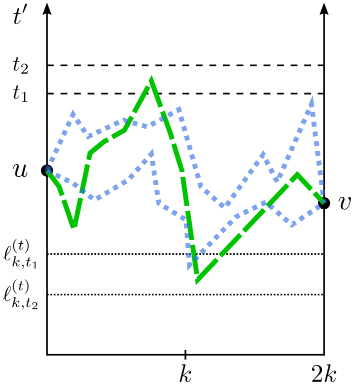

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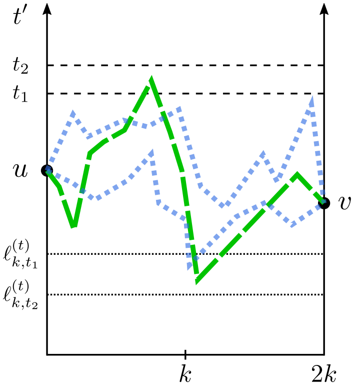

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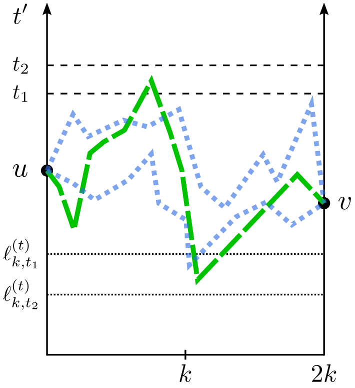

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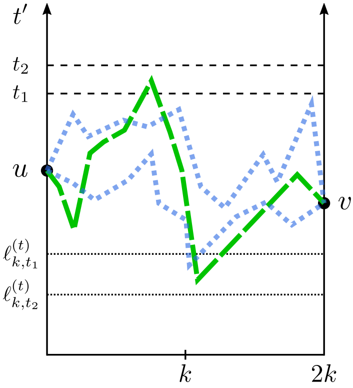

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Upper bound

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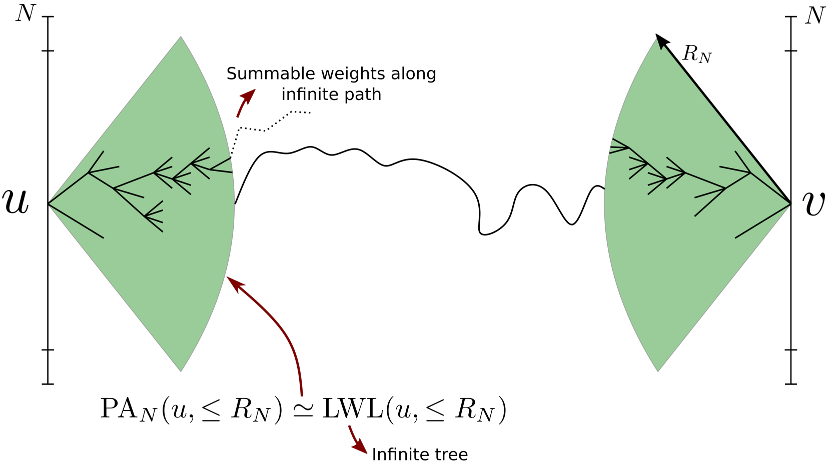
- We will use this info as a black box.

Proof outline for fixed N

y-axes=arrival time of vertices; x-axis: graph distance. LWL=limit distribution of the trees, depth R_N (large constant).



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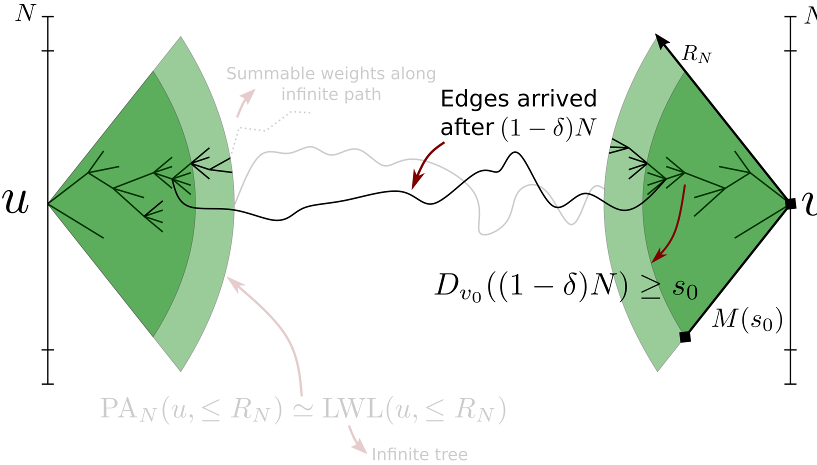

Building the connecting path

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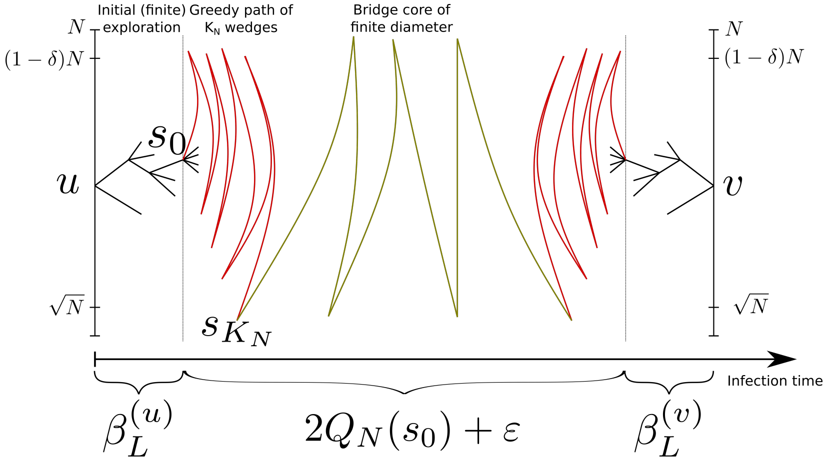
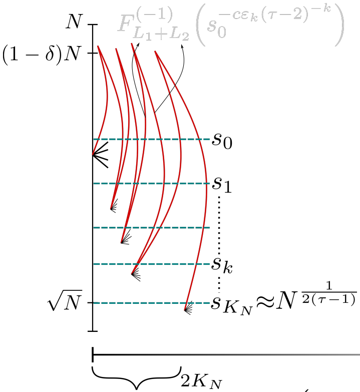


Greedy path to the core

Core: vertices born before time $\sqrt{N}$. Core has bounded diameter.



Careful choice of the sequence $(s_k)_{k\geq 1}$ yields
- Short path to dense inner core

$$K_N \leq \frac{\log\log(N)}{|\log(\tau-2)|} + C$$

- From any vertex in Layer k, there are at least $s_k^{\varepsilon_k}$ young wedges to Layer $k+1$

- "Young" edges ensures independent weights w.r.t. to weighted LWL

- Minimum of i.i.d. weighted wedges

$$\min_{j\leq M}\{L_1^{(j)} + L_2^{(j)}\} \approx F^{(-1)}_{L_1+L_2}(1/M)$$

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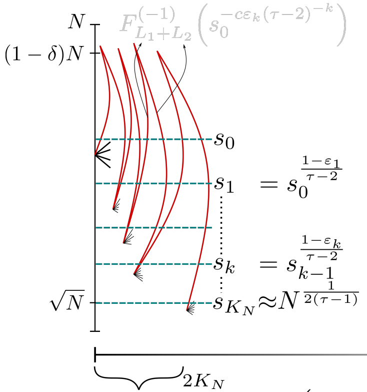

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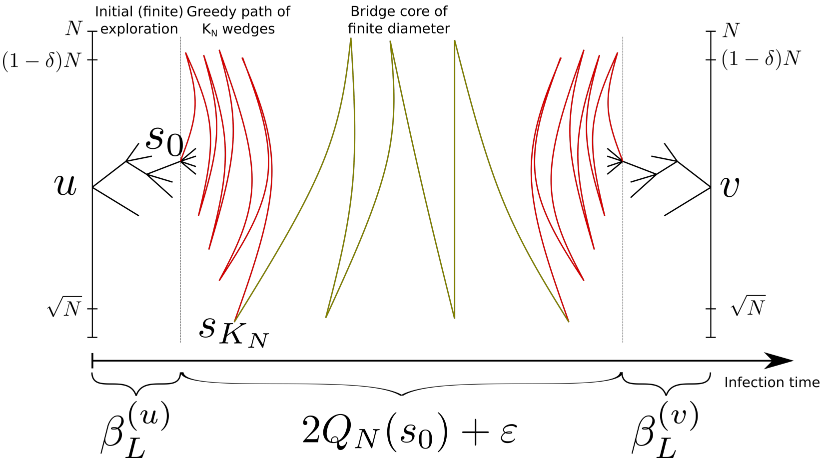
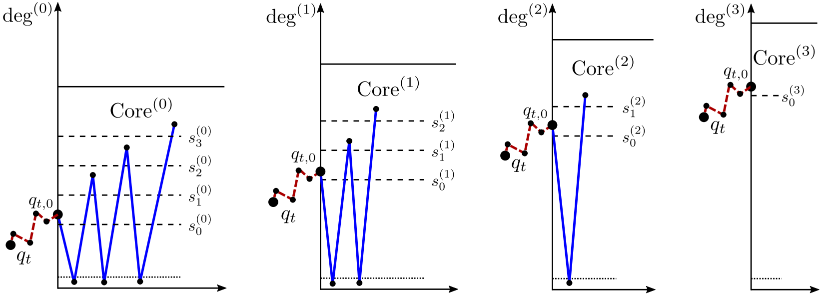
Dense core has small diameter



Extension to the growing network

- We need to track the degree-growth of the vertex that starts the wedging procedure (via a Móri martingale).



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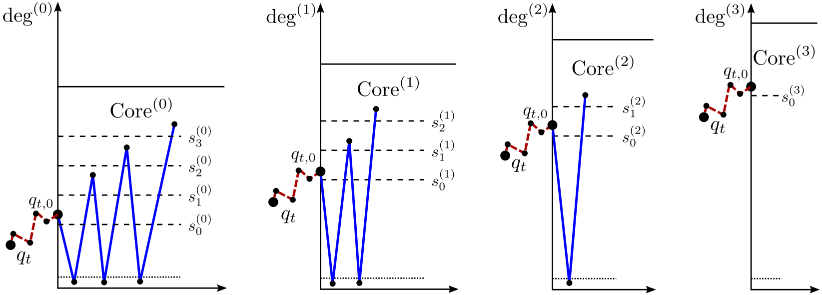

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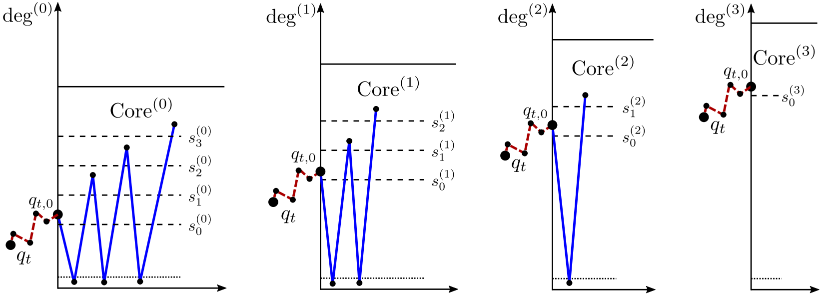
- Careful union bound: summing error probs only where the event changes



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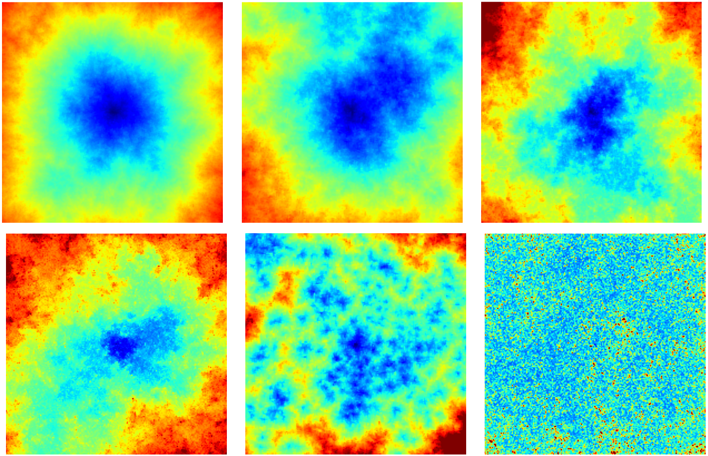
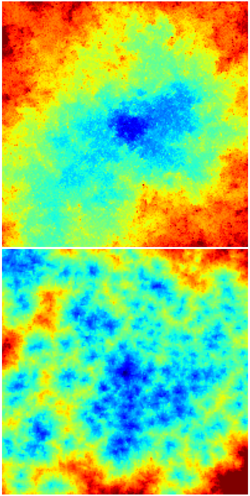


Figure: Six instances of an infection spreading on a two-dimensional spatial scale free networks with different parameters.

Spreading processes on networks



Susceptible-Infected model:

- At time $t = 0$ the source node is infected, all other nodes susceptible.
- if, on an edge $\{u, v\}$, u is infected and v is not, then v becomes infected after a random iid transmission delay $L_{(u,v)}$.

The epidemic curve

The function that counts the total number of infected nodes before time t:

$$I(t) = \#\{\ \textit{infected nodes before time } t\}$$

Weighted-network point of view

Pre-sampling all randomness

Add *iid* weights from distribution L to existing edges.

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$$I(t) = \#\{v : d_L(u,v) \le t\}.$$

Weighted distances in PAMs

$L \equiv 1$: graph distance

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Weighted distances on PA

Theorem (Explosion in PA)

Jorritsma, K, '20

Consider PA with $\tau \in (2,3)$, and U_N, V_N two typical vertices in the giant component of PA_N. Then if

$$E(L) := \sum_{k=1}^{\infty} F_L^{(-1)}\left(1/\mathrm{e}^{\mathrm{e}^k}\right) < \infty,$$

then

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If $E(L) = \infty$ then

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From small to mini-worlds

- $E(L) = \infty$: fluctuations are tight in most cases. Lower tightness always, upper tightness under technical condition that holds for most Ls.

$$E_2(L) = \sum_{k=1}^{\infty} \frac{1}{k} F^{(-1)}_{L-\inf \mathrm{supp} L}\left(1/\mathrm{e}^{\mathrm{e}^k}\right) < \infty$$

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- Fix your favorite $1 \ll g(N) = O(\log\log N)$. Then one can construct a distribution L such that
$$d_G^{(N)}(U_N, V_N) = g(N) + O_{\mathbb{P}}(1).$$

Shrinking of weighted distances when $I(L) = \infty$

For graph distances

$$\sup_{t' \geq t} \left(d_G^{(N')}(U_N, V_N) - 2\Big(\log\log N - \log(1 \vee \log(N'/N)) \Big) \cdot \frac{2}{\log(\tau - 2)|} \vee 1 \right) = O_{\mathsf{P}}(1).$$

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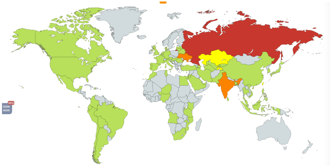
with $K_{N,N'} = \Big(\log\log N - \log(1 \vee \log(N'/N))\Big) \cdot \frac{2}{\log(\tau-2)|} \vee 1$

What is explosion?

On infinite networks

A spreading process is explosive on an infinite network if $I(t) = \infty$ for some $t < \infty$.



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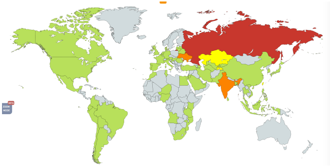
A *positive fraction* of the nodes is infected within *constant time*, *no matter how large the network is*.



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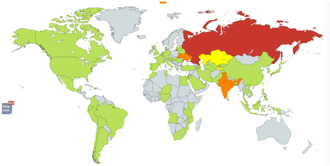

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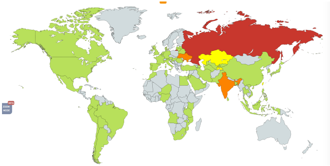

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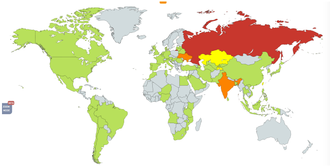

Explosion in PA

$$d_G^{(N)}(U_N, V_N) \xrightarrow{d} \beta_L = Y_1 + Y_2$$

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A deterministic curve with a random constant shift.

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A deterministic curve with a random constant shift.

Explosion in PA

$$d_G^{(N)}(U_N, V_N) \xrightarrow{d} \beta_L \ = Y_1 + Y_2$$

Then

$$\begin{aligned} I(t) &= \frac{1}{N}\#\{v : d^{(N)}(U_N, v) \le t\} \\ &= \frac{1}{N}\sum_{v \in \mathbf{PA}_N} \mathbb{1}_{\{d_G(U_n, v) \le t\}} \\ &= \mathbb{P}_{V_N}(d_G(U_n, V_n) \le t \mid U_n) \\ &\xrightarrow{d} \mathbb{P}(Y_2 \le t - Y_1 \mid Y_1) = g(t - Y_1) \end{aligned}$$

A deterministic curve with a random constant shift.