A randomized k-centrality measure & applications to Networks Node Immunization

Luca Avena (Leiden, The Netherlands)

A day on random graphs, TorVergata, 30/05/2022

joint with **Michael Emmerich, Alexandre Gaudillière and Irina Gurewitsch**

Outline

1. **Compartmental models on networks**
2. **Multiple-node immunization**
3. **Our method: rooted forests & randomized k-centrality**
4. **Experiments: the geometry of contagion**

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Multiple-node immunization & max eigendrop

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- **Problem not well-posed:** 1) limited rigorous analysis 2) heuristics ideas based on simulations with lots of other problems 3) choosing maximal degree nodes is not justified already for $k = 1$.
- **Max eigendrop of adjacency:** **since $r_c \geq 1/\lambda_{max}$,**
  good strategy: look for sets of $k-$nodes from which after removal the reduced network has minimal largest eigenvalue.
  Equivalently: find $\mathcal{S} \subset \mathcal{V}$, with $|\mathcal{S}| = k$ maximising

$$\Delta_k \lambda(\mathcal{S}) := \lambda_{max}(\mathcal{V}) - \lambda_{max}(\mathcal{V} \setminus \mathcal{S}) = \mathbf{k - eigendrop}$$

Computational issues with max eigendrop:

- No poly exact algorithm for such an optimization problem.
- **Chen Chen et al. (2016)** proved **approx. equivalence with so-called Shield-Value optimization which is NP complete**.

Max k-eigendrop, ShieldValue & Netshield (Chen Chen et al. (2016))

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- **Shield Value** : for $\mathcal{S} \subset \mathcal{V}$, with $|\mathcal{S}| = k$ define

$$SV_k(\mathcal{S}) := 2\lambda_{max} \sum_{x \in \mathcal{S}} u^2(x) - \sum_{x,y \in \mathcal{S}} a_{x,y} u(x) u(y)$$

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with u principal eigenvector ($\mathcal{A}u = \lambda_{max} u$).

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$$\Delta_k \lambda(S) \approx SV_k(\mathcal{S}).$$

- **Maximising shieldvalue is NP-complete problem.**

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Our randomized immunization method:

Random Rooted Spanning Forests

&

randomized k-centrality

Graph Laplacian and associated RW

Weighted Directed Network: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$, with $|\mathcal{V}| = n$ and weighted adjacency $\mathcal{A}_w := (a_{x,y})_{x,y \in \mathcal{V}}$ s.t. $a_{x,y} = w(x,y)\mathbb{1}_{\{x \neq y\}}$.

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It is the $n \times n$ matrix $-\mathcal{L}_w$ with :

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In particular, $\mathcal{L}_w = \mathcal{A}_w - \mathcal{D}_w =$ (weighted) "adjacency" $-$ "degree"

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Given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$, and a parameter $q > 0$, let $\mathcal{R}_q \subseteq \mathcal{V}$ be **a random subset of nodes** with law characterized by:

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Forest Space: for $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$, set

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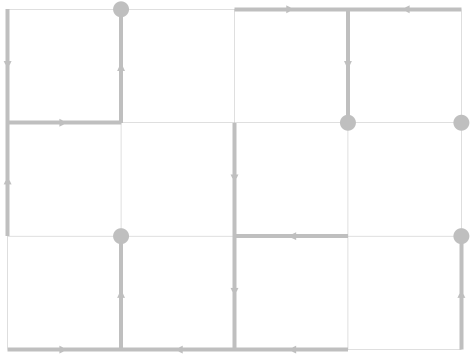

A spanning rooted forest on a square grid.

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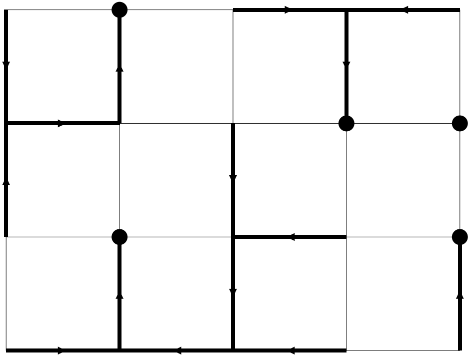

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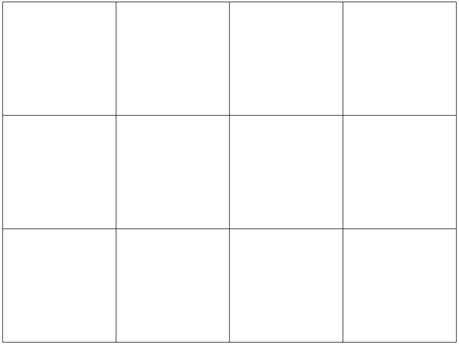

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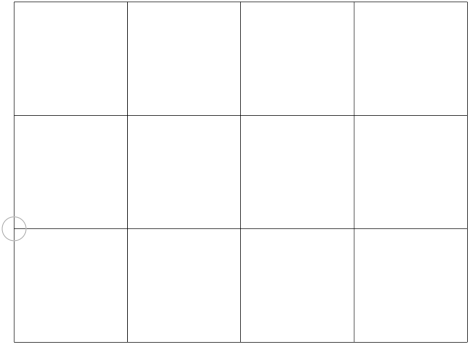

Build dynamically a rooted spanning forest
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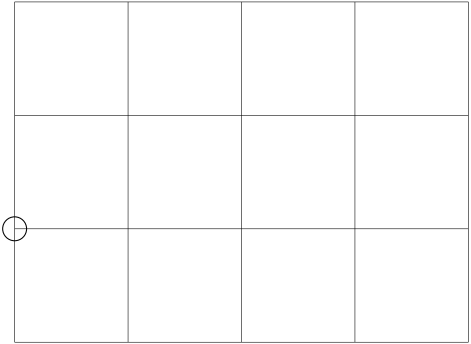

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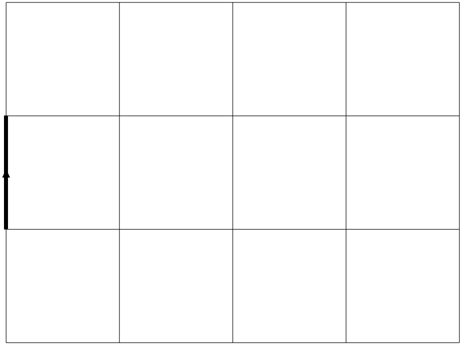

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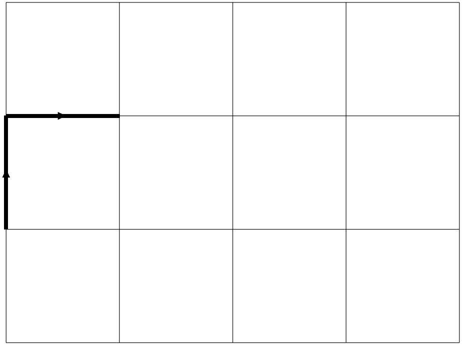

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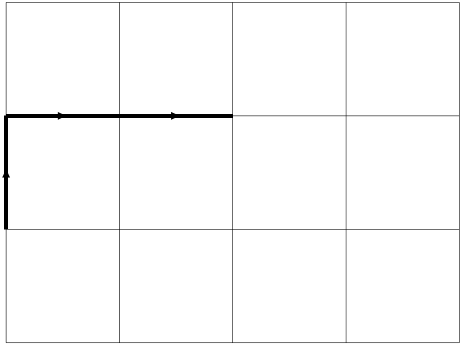

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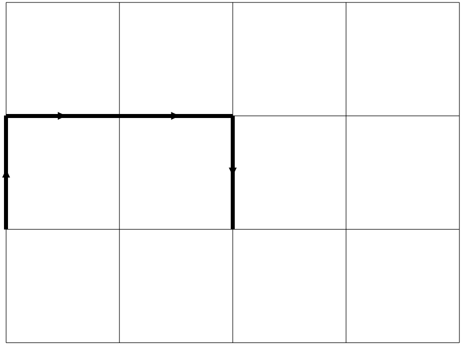

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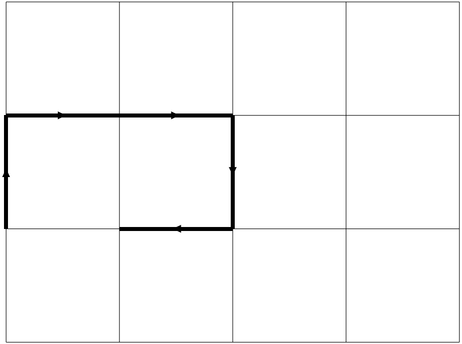

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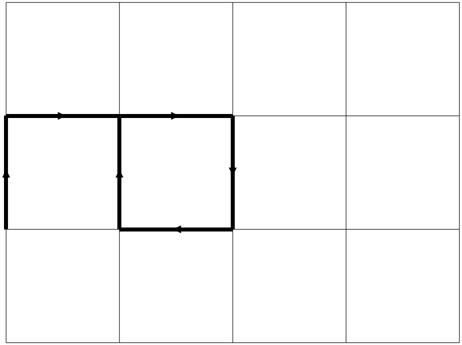

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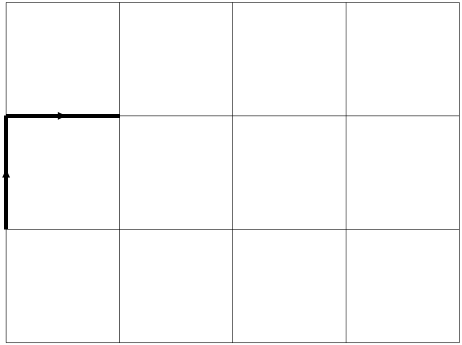

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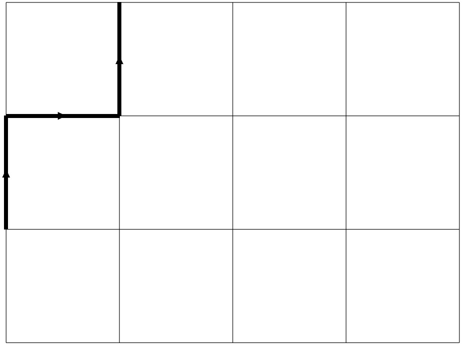

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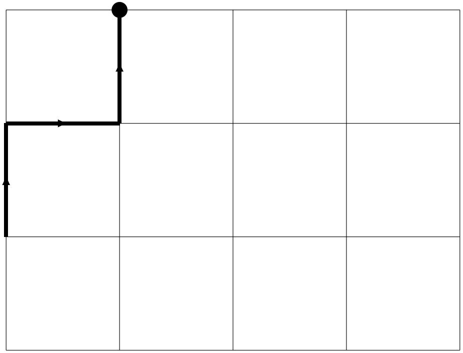

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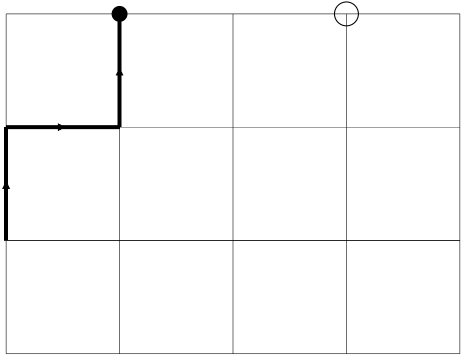

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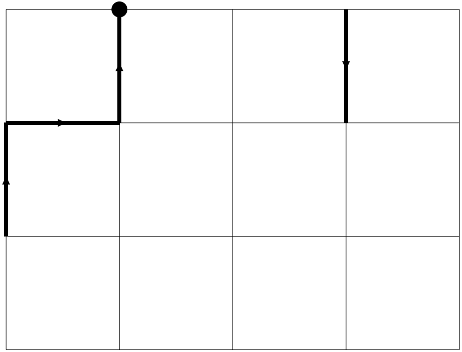

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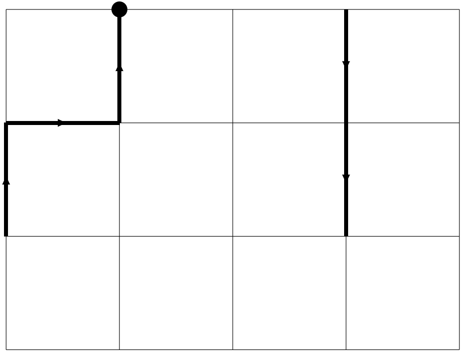

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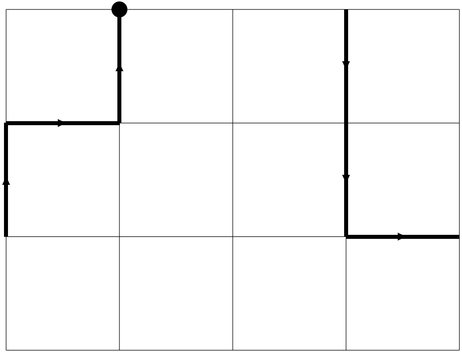

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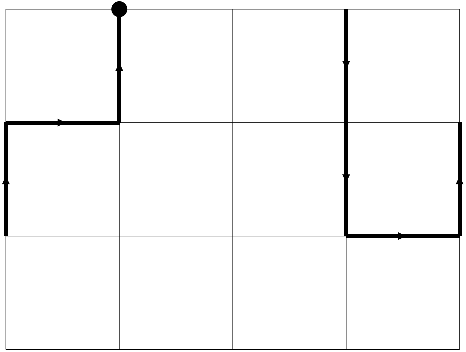

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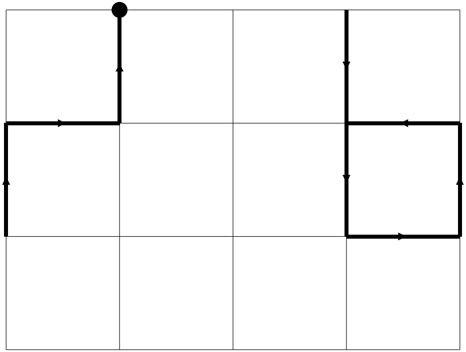

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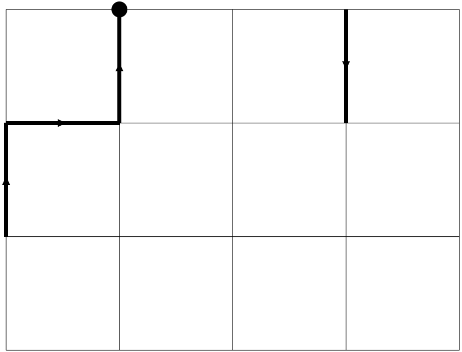

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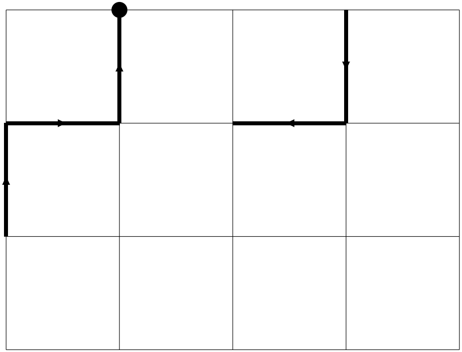

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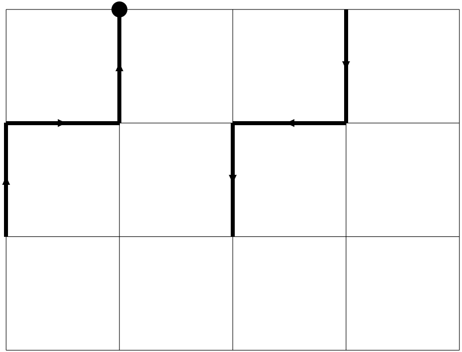

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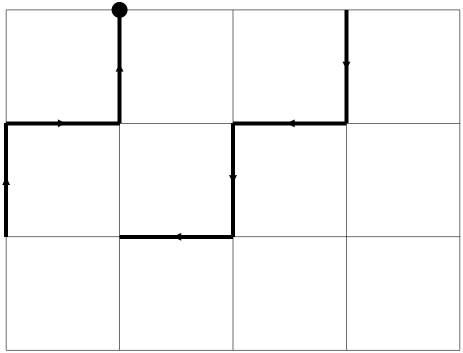

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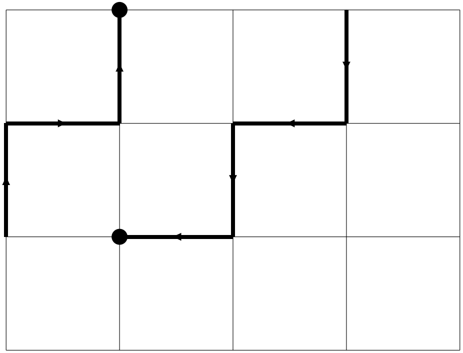

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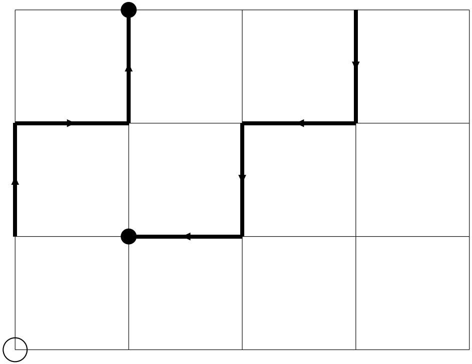

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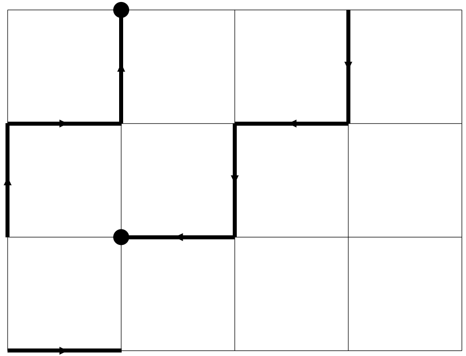

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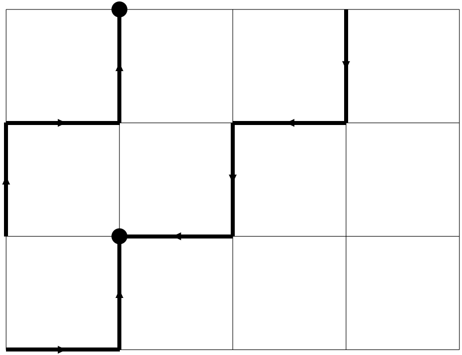

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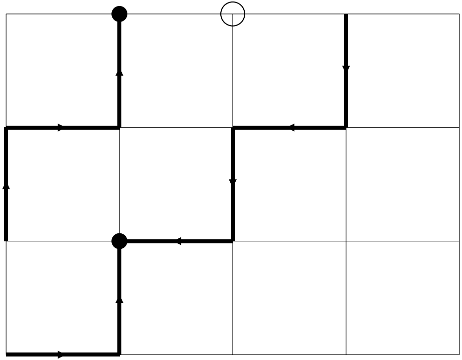

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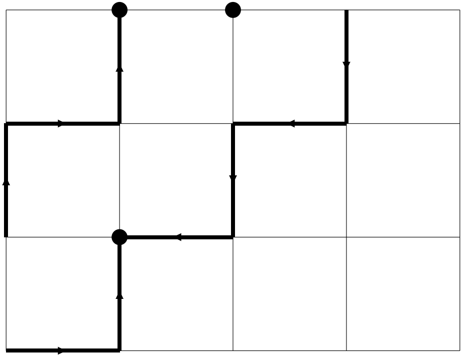

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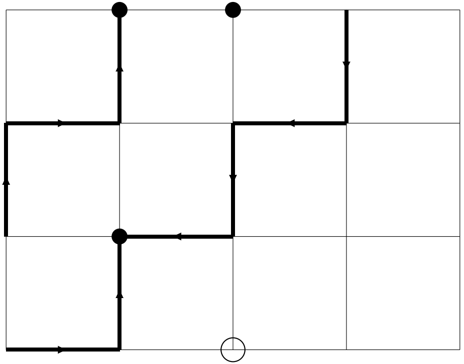

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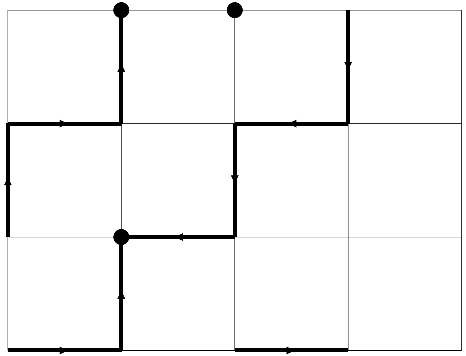

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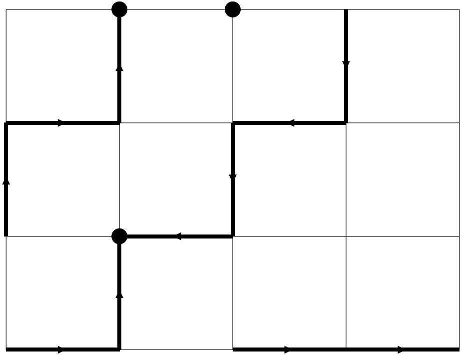

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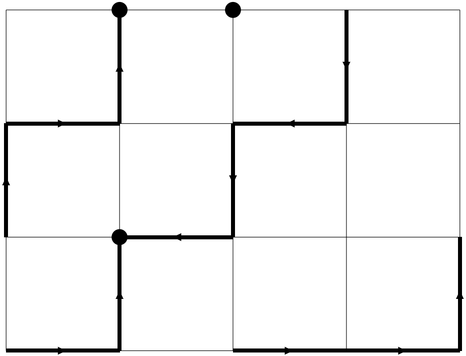

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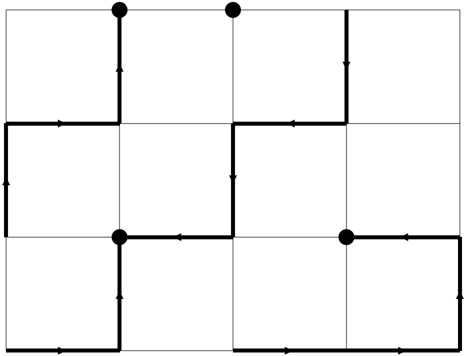

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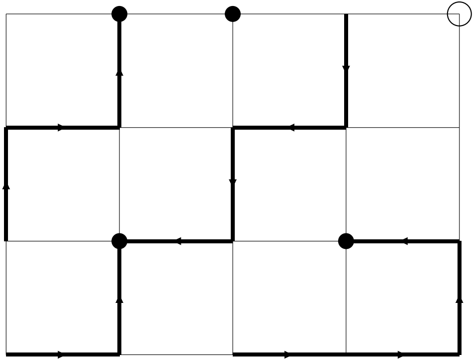

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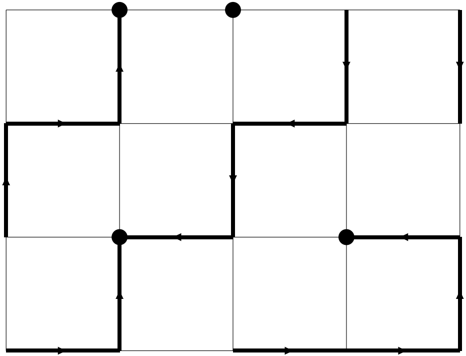

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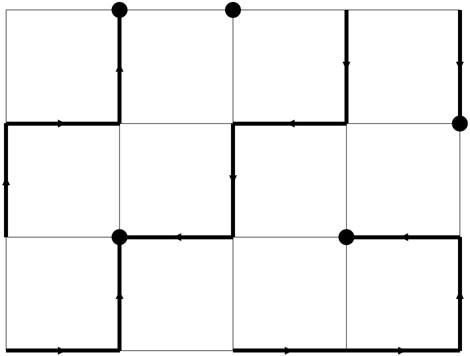

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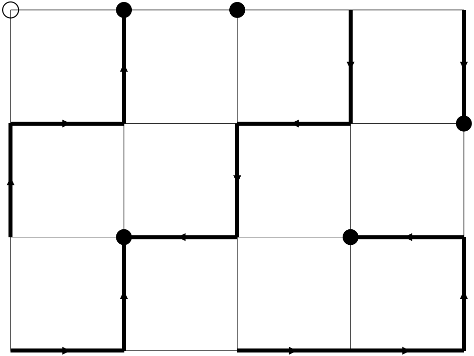

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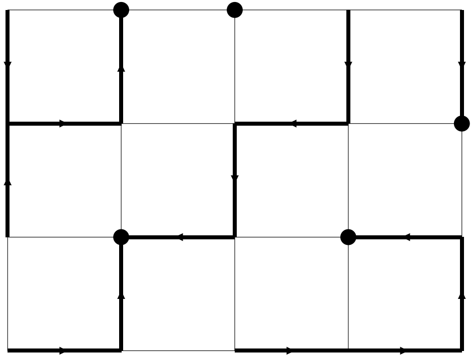

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Given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$, consider the set of Roots $\mathcal{R}_q$ with kernel K_q. Then **its cardinality** *is a* **non-homogeneous Binomial** *:*

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Forest immunization & the geometry of contagion in action:

a few illustrative experiments.

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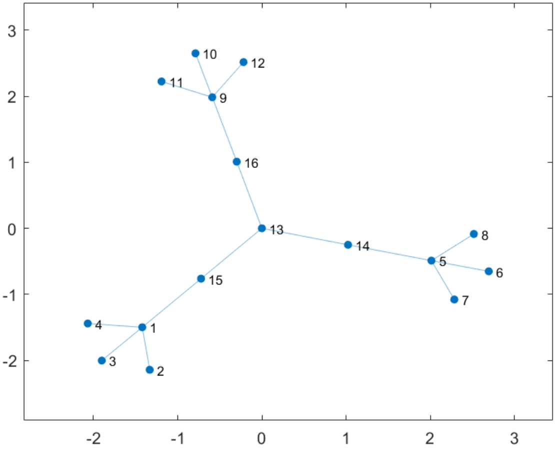

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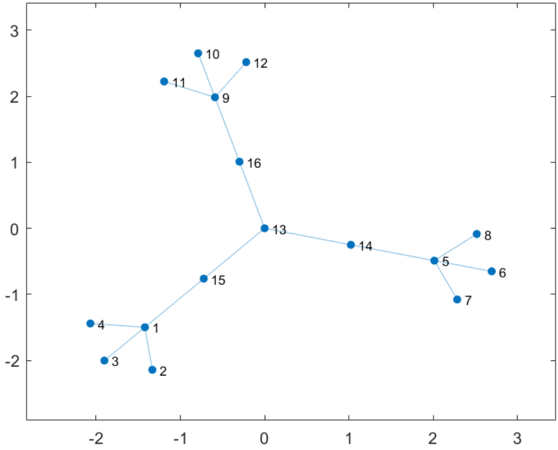

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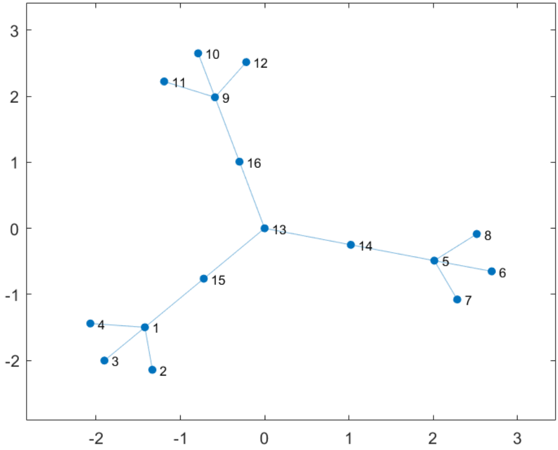

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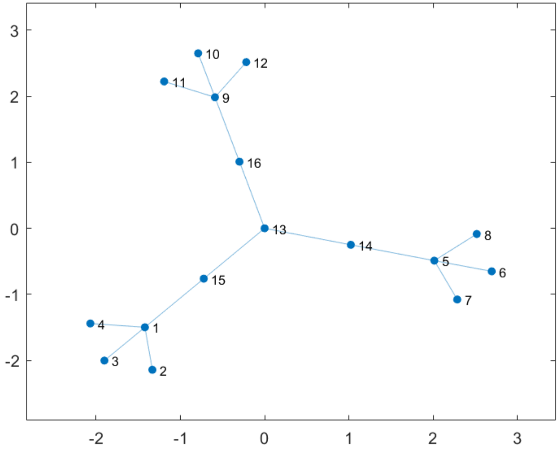

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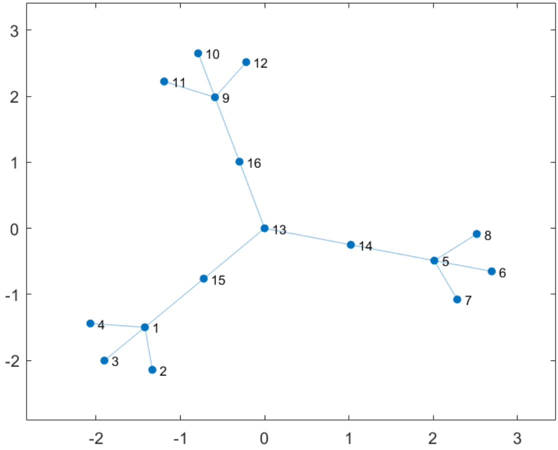

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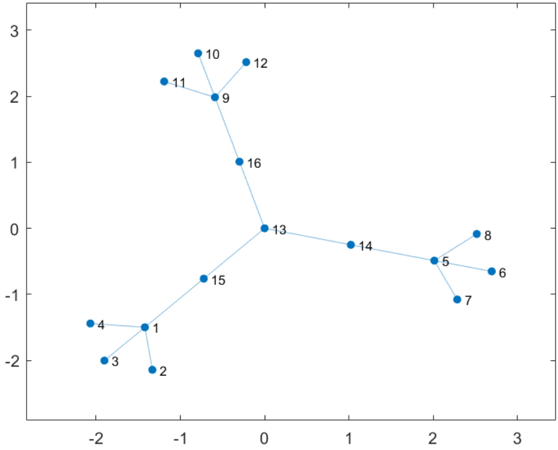

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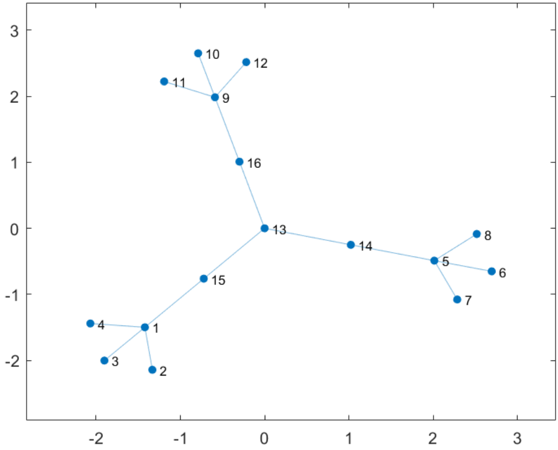

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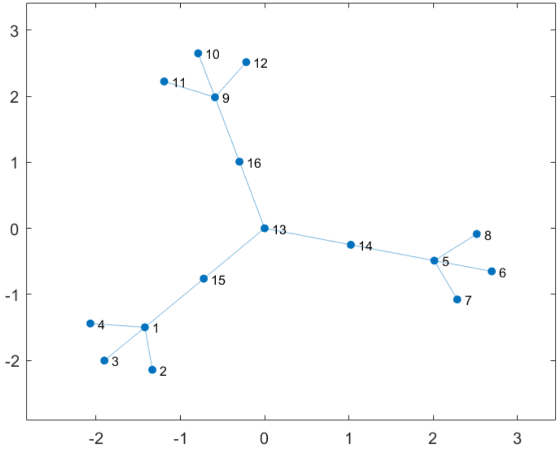

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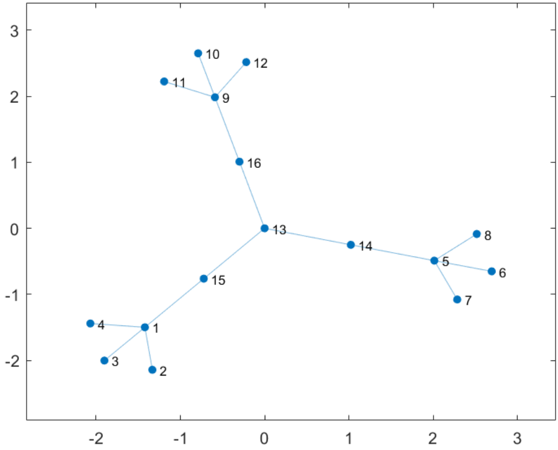

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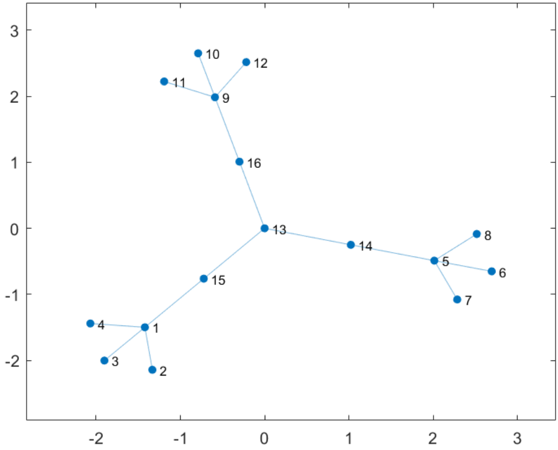

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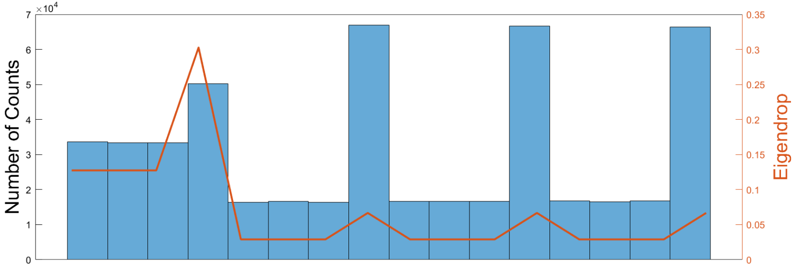

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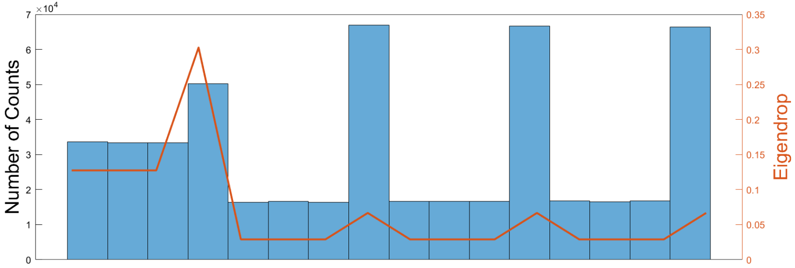

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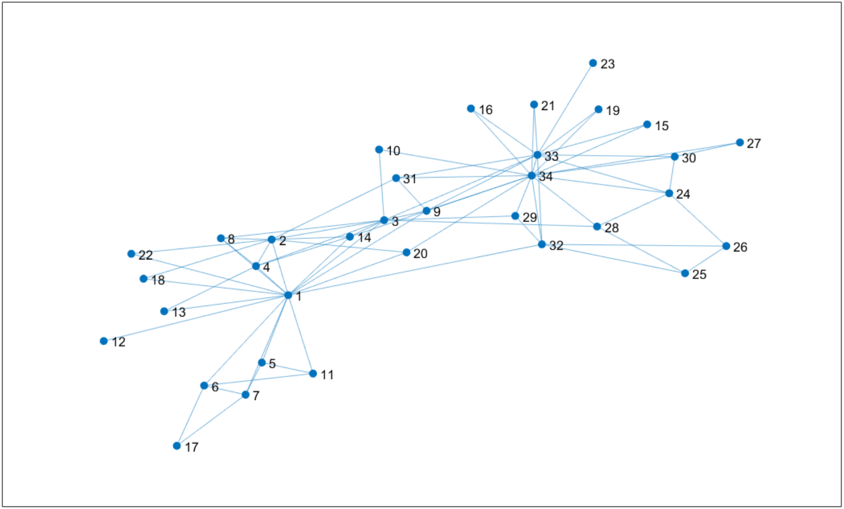

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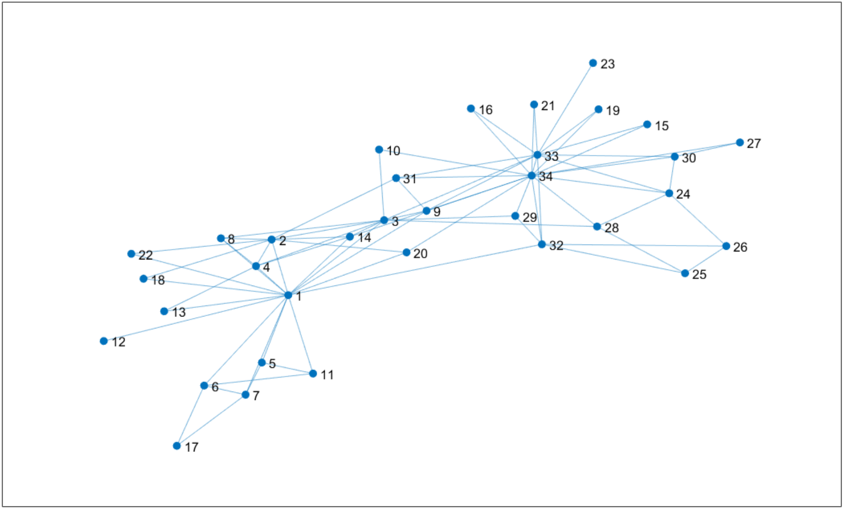

President $\{1\}$ Vs Instructor $\{34\}$ led to a split into 2 groups.

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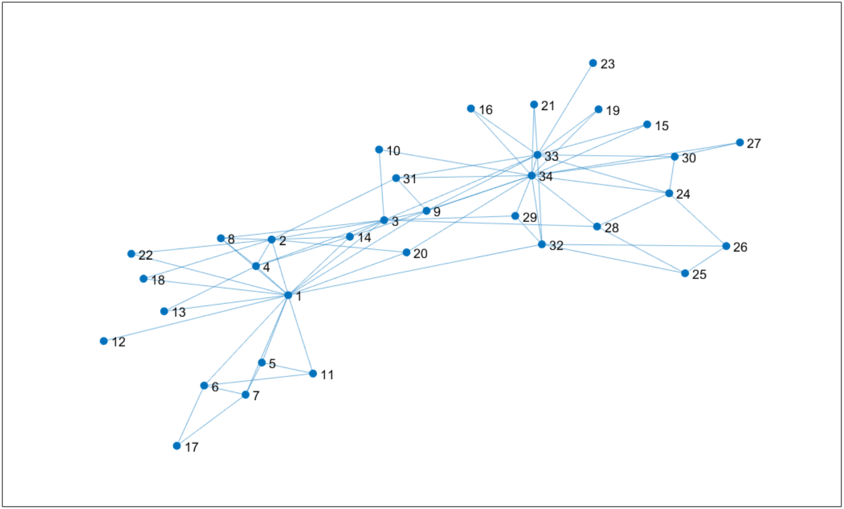

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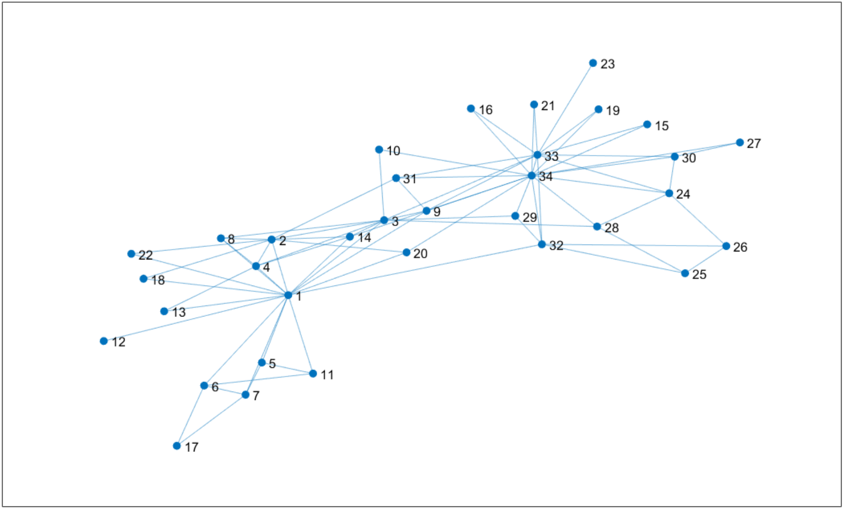
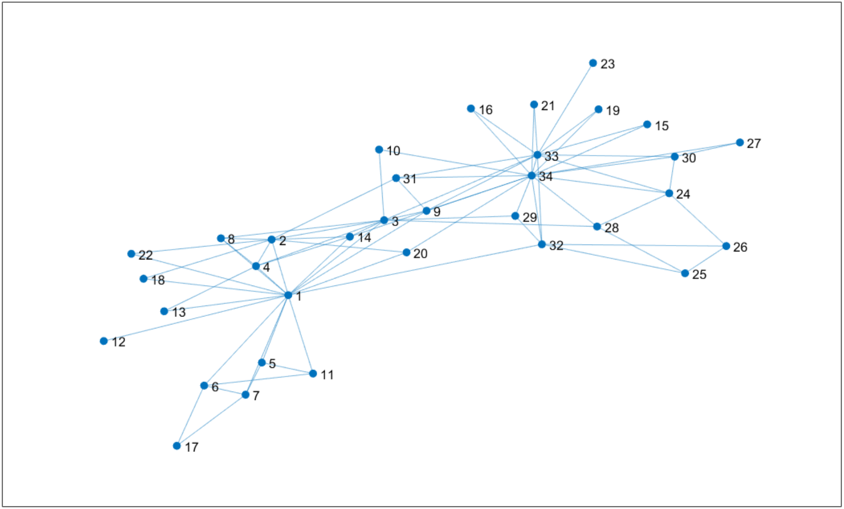


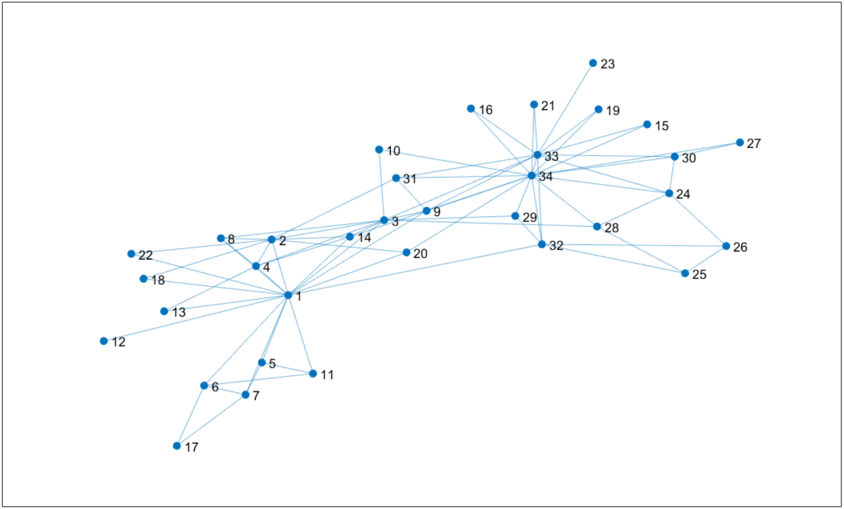
$\Rightarrow$ best node is $\{3\}$: **neither the president nor the instructor.**

The Karate club: $k = 1$



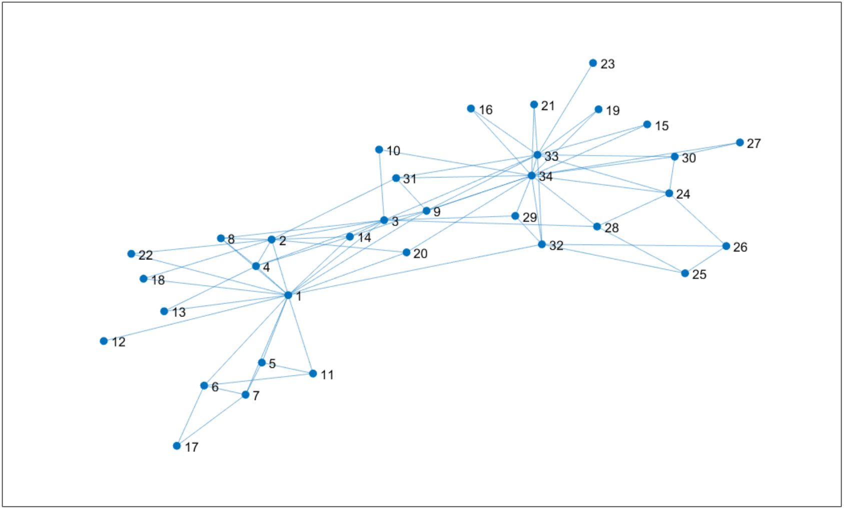
$\Rightarrow$ best node is $\{3\}$: **neither the president nor the instructor.**

The Karate club: $k = 2$



$\Rightarrow$ best pair is $\{1, 34\}$: **the president and the instructor.**

The Karate club: $k = 3$



$\Rightarrow$ best triple is $\{1, 3, 34\}$: **"the fighters and the philanthropist".**

The Karate club: $\mathcal{R}_q^{\mathcal{C}}$ for $k = 1$

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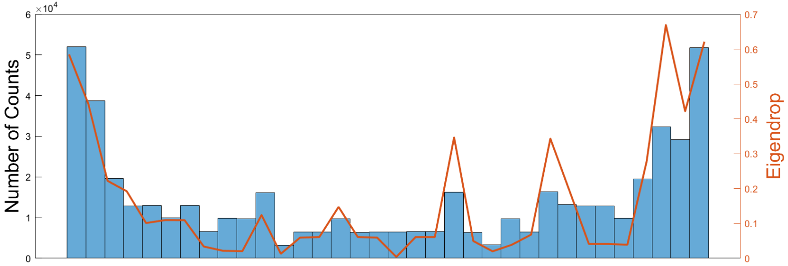
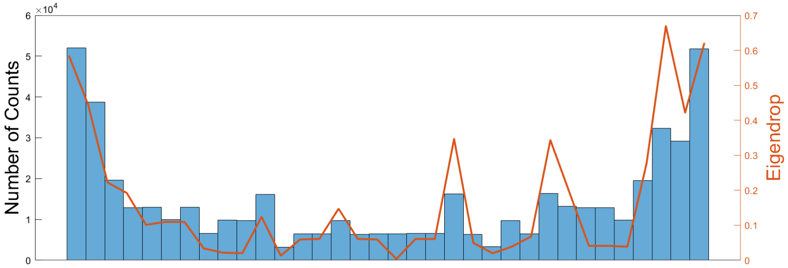


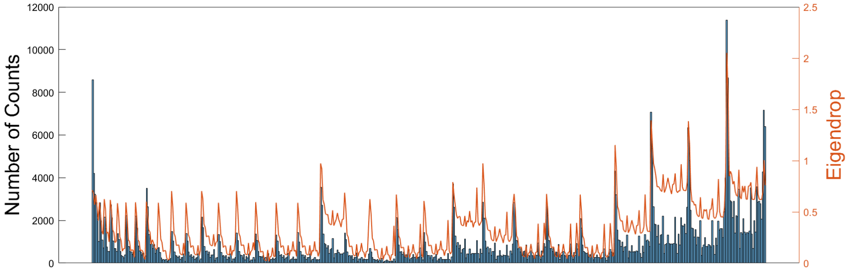
$\mathcal{R}_q^{\mathcal{C}}$ **histogram**, **Eigendrop profile**

The Karate club: $\mathcal{R}_q^{\mathcal{C}}$ for $k = 1$



$\mathcal{R}_q^{\mathcal{C}}$ **histogram**, **Eigendrop profile**

The Karate club: $\mathcal{R}_q^{\mathcal{C}}$ for $k = 2$



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Conference Interaction: a weighted example

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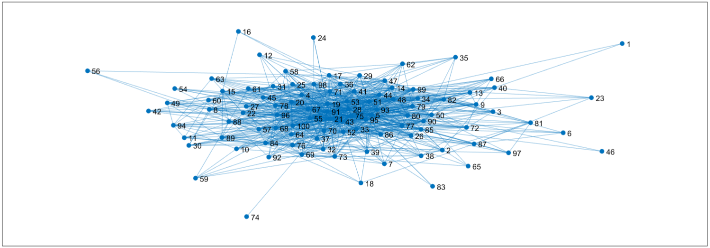
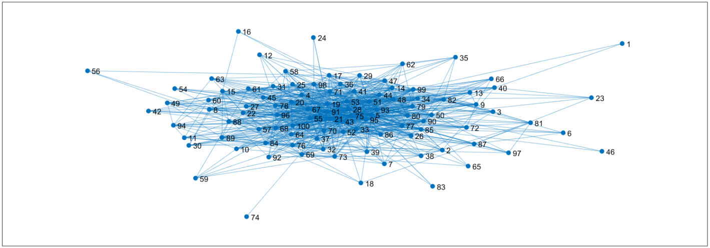


$|\mathcal{V}| = 10^2$ individuals, $\quad |\mathcal{E}| \approx 10^3$ interactions in one day,

On such large non-regular networks benchmarks (and similar ones, e.g. airport, etc...) the **forest-immunization can find better solutions than Nethsield in comparable running time.**

Conference Interaction: a weighted example



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References & THANK YOU

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