TWISTED AKEMANN - OSTRAND PROPERTY FOR $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$ RELATIVE TO $\mathrm{PSL}_2(\mathbb{Z})$ AND ESSENTIAL SPECTRUM OF HECKE OPERATORS ON MAASS FORMS

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ABSTRACT. We generalize the Akemann - Ostrand theorem for $\mathrm{PSL}_2(\mathbb{Z})$, to the case of the partial transformations action of $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])\times\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])^{\mathrm{op}}$, by left and and right multiplication on $\mathrm{PSL}_2(\mathbb{Z})$, in the presence of a twisting cocycle.

0. INTRODUCTION

In this paper we consider a generalization of the Akemann - Ostrand theorem ([1]) for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, to the case of the partial transformations action of

$$\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \times \mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])^{\mathrm{op}},$$

by left and and right multiplication on $\mathrm{PSL}_2(\mathbb{Z})$.

Recall that Akemann - Ostrand property (to which we will refer in the sequel as to the AO property) for the free group F_N, $N \geq 2$ asserts ([1]) the fact that the C^* - algebra, generated in $\mathcal{B}(l^2(F_N))$, simultaneously by the C^* - algebras $C^*_\lambda(F_N)$, $C^*_\rho(F_N)$ that are generated by the left and respectively, the right convolution operators with elements in F_N, is isomorphic, modulo

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the ideal $\mathcal{K}(l^2(F_N))$ of compact operators, to the minimal C^* - tensor product $C^*_{\rm red}(F_N) \underset{\min}{\otimes} C^*_{\rm red}(F_N^{\rm op}) \cong C^*_{\rm red}(F_N \times F_N^{\rm op})$ of the reduced group C^* - algebras associated to F_N. Here, by definition the reduced group C^* - algebra $C^*_{\rm red}(\Gamma)$ of a discrete group Γ is $C^*_\lambda(\Gamma)$.

The Akemann - Ostrand property has been widely extended. G. Skandalis proved ([35]) that the same result remains true for lattices in semisimple Lie groups of rank 1. Using amenable actions techniques ([3]), Gunter and Higson ([13]) and then Ozawa ([27]) have further extended this result, to large classes of hyperbolic groups.

The key in Ozawa's approach in proving the AO property for a discrete group Γ is the amenability (see e.g [3]) of the action of $\Gamma \times \Gamma^{\rm op}$ on the boundary $\partial(\beta\Gamma)$ of the Stone Cech compactification of Γ, viewed as a discrete set. This stronger property for a group Γ is called ([27],[9]) property $\mathcal{S}$.

Consider the canonical representation, which we denote by $\pi_{\rm Koop}$, of the crossed product C^* - algebra $C^*((\Gamma \times \Gamma^{\rm op}) \ltimes C(\partial(\beta\Gamma)))$ into $\mathcal{B}(l^2(\Gamma))$. Let

$$\pi_Q : \mathcal{B}(l^2(\Gamma)) \to \mathcal{Q}(l^2(\Gamma)) = \mathcal{B}(l^2(\Gamma))/\mathcal{K}(l^2(\Gamma))$$

be the projection onto the Calkin algebra ([10]). The $\mathcal{S}$ property ([26]) implies that the representation

$$\pi_Q \circ \pi_{\rm Koop} : C^*((\Gamma \times \Gamma^{\rm op}) \ltimes C(\partial(\beta\Gamma))) \to \mathcal{Q}(l^2(\Gamma)),$$

factorizes to a representation of the reduced C^* - algebra

$$C^*_{\rm red}((\Gamma \times \Gamma^{\rm op}) \ltimes C(\partial(\beta\Gamma))).$$

In this paper we extend the Akemann - Ostrand property in the following sense. Let Γ be the modular group $\mathrm{PSL}_2(\mathbb{Z})$. Let $G = \mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$, p a prime ≥ 2.

We denote by G_p, G_r, K the groups $PSL(2,\mathbb{Q}_p)$, $PSL(2,\mathbb{R})$, $PSL(2,\mathbb{Z}_p)$ and by $\widetilde{G}_p$, $\widetilde{G}_r, \widetilde{K}$ their SL_2 versions. Note that second series of groups are a central series by $\mathbb{Z}_2$ of the previous one. Let χ be the non-trivial character of $\mathbb{Z}_2$, Let s be the diagonal matrix with entries -1, which is an element of each of those groups and let p_χ be the projection $\frac{1-s}{2}$ which belongs to the C*-algebra (or multiplier algebra) of each of the previous groups. Let ϵ be the (non canonical) 2 cocycle with values in $\mathbb{Z}_2$, corresponding to the projective representation π_{13} in the analytic series of G_p (see e.g. [17]).

In this paper we are interested in the twisted C* product algebra

$$\mathcal{A} = C^*((G \times G^{\mathrm{op}}) \ltimes_{\epsilon,\epsilon} C(K)).$$

In fact the above crossed product is a grupoid crossed product as the action of $G \times G^{\mathrm{op}}$ on K is a partial action (see the Preliminaries section). The algebra $\mathcal{A}$ has a canonical representation into $B(l^2(\Gamma))$. To avoid the complication of a choice of the two cocycle ϵ we use the technique in [23] and identify $l^2(\Gamma)$ with $p_\chi l^2(\widetilde{\Gamma})$, and consider

$$\widetilde{\mathcal{A}} = C^*((\widetilde{G} \times \widetilde{G}^{\mathrm{op}}) \ltimes C(\widetilde{K})).$$

Then $\widetilde{\mathcal{A}}$ is canonically represented into $B(l^2(\widetilde{\Gamma}))$ and then

$$\mathcal{A} = p_\chi \widetilde{\mathcal{A}} p_\chi \subseteq B(p_\chi l^2(\widetilde{\Gamma})) \cong B(l^2(\Gamma)). \tag{1}$$

Let π_Q be the projection onto the Calkin algebra $Q((l^2(\Gamma)) = B(l^2(\Gamma))/K(l^2(\Gamma))$ The main theorem of the second section is

Theorem 1. *Let Π_Q be the representation of the algebra $\mathcal{A}$ into $Q((l^2(\Gamma)))$ obtained by composing the representation of $\mathcal{A}$ in formula 1 with π_Q. Then Π_Q factorizes to the reduced crossed product algebra*

$$\mathcal{A}_{\mathrm{red}} = C^*_{\mathrm{red}}((G \times G^{\mathrm{op}}) \ltimes_{\epsilon,\epsilon} C(K)).$$

To prove the theorem we will first prove (while considering the partial action of $G \times G^{\mathrm{op}}$ on K) that

Theorem 2. *The crossed product algebra*

$$\mathcal{B} = C^*((G \times G^{\mathrm{op}}) \ltimes_{\epsilon,\epsilon} C_0(G_r \times K \times {G_r}^{\mathrm{op}}))$$

coincides with the reduced crossed product algebra

$$\mathcal{B}_{\mathrm{red}} = C^*_{\mathrm{red}}((G \times G^{\mathrm{op}}) \ltimes_{\epsilon,\epsilon} C_0(G_r \times K \times {G_r}^{\mathrm{op}})).$$

Moreover the same holds true if $G_r \times K \times {G_r}^{\mathrm{op}}$ *is replaced by* $P^1(\mathbb{R}) \times K \times P^1(\mathbb{R})$.

Theorem 2 will be used to prove Theorem 1 by showing that inside $\ell^\infty(\Gamma)/c_0\Gamma$ there is a $G \times G^{\mathrm{op}}$-equivariant "amenable quotient" of $G_r \times G_p \times {G_r}^{\mathrm{op}}$.

1. Preliminaries on the partial action of G

It is well known ([12]), that Γ is almost normal in G. The almost normal property for the subgroup Γ of G signifies that for all $g \in G$ the subgroup

$$\Gamma_g = g\Gamma g^{-1} \cap \Gamma \subseteq \Gamma, \tag{2}$$

has finite index $[\Gamma : \Gamma_g]$.

The group G acts naturally, by conjugation, by partial isomorphisms, on Γ. Indeed for $g \in G$, the conjugation by g on G, will restrict to a partial isomorphism

$$\Delta(g) : \Gamma_{g^{-1}} \to \Gamma_g. \tag{3}$$

It is well known that in the case of the example of the modular group, that we are considering in this paper, we have that $[\Gamma : \Gamma_g] = [\Gamma : \Gamma_{g^{-1}}]$, for all $g \in G$. We consider the family of maximal normal subgroups Γ_g^0 contained in Γ_g.

Let K be the compact space obtained as the inverse limit of the finite coset spaces Γ/Γ_g^0 as $g \to \infty$. Then K is the totally disconnected subgroup

$PSL(2,\mathbb{Z}_p)$, with Haar measure μ_K defined by the requirement that the compact set corresponding to the closure of a coset $s\Gamma_g, s \in \Gamma$ in the profinite topology, has Haar measure equal to $\frac{1}{[\Gamma : \Gamma_g]}$, $g \in G$.

The condition that $[\Gamma : \Gamma_g] = [\Gamma : \Gamma_{g^{-1}}]$, implies that the partial transformation $\Delta(g)$, introduced in formula 3 induced by conjugation with $g \in G$, preserves the Haar measure μ_K on K.

There is a natural action of $G \times G^{\mathrm{op}}$ on K. An element $(g_1, g_2) \in G \times G^{\mathrm{op}}$ acts by partial transformations on K, by mapping, $k \in K$ into $g_1 k g_2^{-1}$, if the later element also belongs to K. Thus, the domain of (g_1, g_2), as a partial transformation on K, is

$$\mathcal{D}_{(g_1,g_2)} = \{k \in K \mid g_1 K g_2^{-1} \in K\} = K \cap g_1^{-1} K g_2 = K \cap g_1^{-1} K g_1 (g_1^{-1} g_2).$$

We use the notation $K_g = K \cap gKg^{-1}$. In our construction this is the closure, in the profinite completion, of Γ_g.

2. Proof of Theorem 2

To prove theorem 2 we will consider first crossed product actions with the left and right actions of the groups G_p, G_r. Consider the algebra

$$\widetilde{\mathcal{C}} = [p_\chi \otimes (p_\chi)^{\mathrm{op}}][C^*((\widetilde{G}_p \times \widetilde{G}_p^{\mathrm{op}}) \ltimes C_0(\widetilde{G}_p))]. \tag{4}$$

Here $(p_\chi)^{\mathrm{op}}$ is the projection corresponding to the generator s of $\mathbb{Z}_2$ viewed as an element in ${G_p}^{\mathrm{op}}$, and $p_\chi \otimes (p_\chi)^{\mathrm{op}}$ is a central projection.

If choose a Borel lifting of G_p to $\widetilde{G}_p$ this would induce a two cocycle $\widetilde{\epsilon}$ on G_p and the algebra $\widetilde{\mathcal{C}}$ is isomorphic to the skewed crossed product algebra

$$\mathcal{C} =_{\mathrm{red}} C^*((G_p \times {G_p}^{\mathrm{op}}) \ltimes_{\widetilde{\epsilon},\widetilde{\epsilon}} C_0(G_p)).$$

We prove first the following lemma

Lemma 3. *The crossed product C*- algebra $\widetilde{\mathcal{C}}$ introduced in formula 4 coincides with the reduced crossed product algebra*

$$\widetilde{\mathcal{C}}_{\rm red} = [p_\chi \otimes (p_\chi)^{\rm op}][C^*_{\rm red}((\widetilde{G}_p \times \widetilde{G}_p^{\rm op}) \ltimes C_0(\widetilde{G}_p))].$$

Proof. To simplify the explanation we will use the notation of the ϵ terminology. Thus we have

$$\widetilde{\mathcal{C}} = C^*((G_p \times {G_p}^{\rm op}) \ltimes_{\widetilde{\epsilon},\widetilde{\epsilon}} C_0(G_p)). \tag{5}$$

Let $C^*(G_p, \widetilde{\epsilon})$ be the skewed group C*-algebra of the group G_p with respect the cocycle $\widetilde{\epsilon}$. This is isomorphic to $p_\chi(\widetilde{G}_p)$. We use the approach from the paper [4], Example 7.6. Because in the supplementary series of irreducible unitary representations of $\widetilde{G}_p$, the element s is always mappd into the identity (see chapter 2.3.7 [5] and also [32]) it follows that

$$C^*(G_p, \widetilde{\epsilon}) = {C^*}_{\rm red}(G_p, \widetilde{\epsilon}).$$

The crossed product algebra $\widetilde{\mathcal{C}}$ may be written as the iterated crossed product

$$C^*({G_p}^{\rm op} \ltimes_{\widetilde{\epsilon}} [C^*(G_p \ltimes_{\widetilde{\epsilon}} C_0(G_p))]). \tag{6}$$

By the amenability of the action of G_p on G_p it follows that the inner crossed product algebra $C^*(G_p \ltimes_{\widetilde{\epsilon}} C_0(G_p)$ coincides with the reduced crossed product algebra ${C^*}_{\rm red}(G_p \ltimes_{\widetilde{\epsilon}} C_0(G_p)$ and this in turn coincides with the algebra $K(L^2(G_p, \nu_p))$, where ν_p is the Haar measure on G_p. Because of the observation at the start of the proof and because of the Theorem 7.5 in [4] it follows that also the outer maximal C*crossed product in formula 6 coincides with the reduced C*crossed product. Since both iterated C*-crossed products in formula 6 coincide with the reduced C*-crossed products it follows that the algebra $\widetilde{\mathcal{C}}$ in formula 5 coincides with the reduced crossed product algebra

$${C^*}_{\rm red}((G_p \times {G_p}^{\rm op}) \ltimes_{\widetilde{\epsilon},\widetilde{\epsilon}} C_0(G_p)).$$

□

The next lemma combines the left right action of G_p and G_r.

Lemma 4. *Let* $\mathcal{G} = (G_r \times {G_r}^{\mathrm{op}}) \times (G_p \times {G_p}^{\mathrm{op}})$ *act on*

$$\mathcal{X} = G_r \times (G_r)^{\mathrm{op}} \times G_p. \tag{7}$$

Here $G_p \times {G_p}^{\mathrm{op}}$ *acts by left and right action on* G_p*, and* G_r *and respectively* $(G_r)^{\mathrm{op}}$ *act on* $C_0(G_r)$ *and respectively* $C_0((G_r)^{\mathrm{op}})$*. Then the twisted* *C*-crossed product*

$$C^*(\mathcal{G} \ltimes_{(\epsilon,\widetilde{\epsilon}),(\epsilon,\widetilde{\epsilon})} C_0(\mathcal{X}))$$

coincides with the reduced crossed product

$$C^*{}_{\mathrm{red}}(\mathcal{G} \ltimes_{(\epsilon,\widetilde{\epsilon}),(\epsilon,\widetilde{\epsilon})} C_0(\mathcal{X})).$$

Proof. The crossed product algebra $C^*(\mathcal{G} \ltimes_{(\epsilon,\widetilde{\epsilon}),(\epsilon,\widetilde{\epsilon})} C_0(\mathcal{X}))$ splits as

$$C^*(G_r \ltimes_\epsilon C_0(G_r)) \otimes C^*({G_r}^o \ltimes_\epsilon C_0({G_r}^o)) \otimes C^*(G_p \ltimes_{\widetilde{\epsilon}} C_0(G_p)). \tag{8}$$

By the amenability of the action of G_r on G_r, the first two algebras are nuclear and all the tensor products are minimal tensor products. The third factor in the tensor product, by the previous lemma coincides with the reduced tensor product.

□

Corollary 5. *Let* P *be the quotient* $G_r/AN \cong P^1(\mathbb{R})$*, endowed with the standard action of* G_r*. Let* $\mathcal{Y} = P \otimes G_p \otimes P$*. Let* $\mathcal{G}$ *act on* $\mathcal{Y}$ *as in the previous statement.*

Then the twisted *C*-crossed product*

$$C^*(\mathcal{G} \ltimes_{(\epsilon,\widetilde{\epsilon}),(\epsilon,\widetilde{\epsilon})} C_0(\mathcal{Y}))$$

coincides with the reduced crossed product

$$C^*{}_{\mathrm{red}}(\mathcal{G} \ltimes_{(\epsilon,\widetilde{\epsilon}),(\epsilon,\widetilde{\epsilon})} C_0(\mathcal{Y})).$$

Proof. The proof of the previous statement was using only the amenability of the action of G_r on G_r. Hence the same proof works if replace G_r by P.

$\square$

With this we can conclude the proof of Theorem 2.

Proof. (Theorem 2)

We use the fact the group G is a lattice in $G_{pr} = G_r \times G_p$. Consider the positive definite function associated to the characteristic function of $A \times K \times B$, where A, B are compact subsets of G_r in the twisted Koopmann representation of $C^*((G_{pr} \times {G_{pr}}^{\mathrm{op}}) \ltimes_{(\epsilon,\widetilde{\epsilon}),(\epsilon,\widetilde{\epsilon})} C_0(\mathcal{X}))$ into $B(L^2(\mathcal{X}, \mu))$, where μ is the product Haar measure. The positive definite kernel associated to this vector in the sense of [3] is defined, for $g_1, g_2 \in G_{pr}$, $x, y \in G_r, k \in K$, by the formula

$$F_{A,K,B}((g_1, g_2), (x, k, y)) = \chi_{g_1 A \cap A}(x) \chi_{g_1 K g_2 \cap K}(k) \chi_{B g_2 \cap B}(y).$$

Note that when checking positive definiteness one has to consider the cocycle in the corresponding sums expressing positivity. Because of Corollary 5 this function is a limit, uniformly on compacts of positive definite functions, of compact support in $G_{pr} \times {G_{pr}}^{\mathrm{op}} \times \mathcal{X}$. Because G is a lattice in G_{pr}, the restriction of F to $G \times G^{\mathrm{op}} \times \mathcal{X}$ will also be a limit of positive definite functions (with respect to ϵ) of compact support in $G \times G^{\mathrm{op}} \times \mathcal{X}$. But then any positive definite function on $C^*((G \times G^{\mathrm{op}}) \ltimes_{\epsilon,\epsilon} C_0(G_r \times K \times {G_r}^{\mathrm{op}}))$, which has compact support in the variables corresponding to $\mathcal{X}$, by multiplying with the above approximations will become a limit of positive definite functions of compact support. The general case is obtained by taking limits while increasing the supports in the variables corresponding to $\mathcal{X}$.

$\square$

Similarly because of Corollary 5 we obtain (and also prove the last statement in the proof of Theorem2):

Corollary 6. *The twisted C*crossed product algebra*

$$C^*((G \times G^{\rm op}) \ltimes_{\epsilon,\epsilon} C_0(P \times G_p \times P^{\rm op}))$$

*coincides with the C*reduced crossed product algebra*

$$C^*_{\rm red}((G \times G^{\rm op}) \ltimes_{\epsilon,\epsilon} C_0(P \times G_p \times P^{\rm op})).$$

Proof. The only thing that changes from the previous proof is the fact that this time P carries a natural quasi-invariant measure. Taking the Maharam extension ([21]) reduces the proof to the case of invariant measure. □

3. A $G \times G^{\rm op}$ EQUIVARIANT BOUNDARY INSIDE $\ell^\infty(\Gamma)/c_0(\Gamma)$

Let $\partial\Gamma$ be the (Gromov) boundary of Γ (which is independent on the choice of generators, see e.g. [15]). Let S, T be the standard system of generators of $\Gamma \cong \mathbb{Z}_2 * \mathbb{Z}_3$ and let ν be the homogenous measure on $\partial\Gamma$ associated to this system of generators. Let π_d be the Γ equivariant projection from $\partial\Gamma$ onto $P^1(\mathbb{R})$ constructed in [33], [34] (the Spielberg disconnection). In this section we describe a canonical G action on $\partial\Gamma$ that is G-equivariant. We construct an embedding of $C_0(\partial\Gamma \times \partial\Gamma^{\rm op})$ inside $\ell^\infty(\Gamma)/c_0(\Gamma)$, and prove that the corresponding embedding is $G \times G^{\rm op}$ equivariant with respect to the canonical action of $G \times G^{\rm op}$ on $\ell^\infty(\Gamma)/c_0(\Gamma)$

We divide this in several steps.

Lemma 7. *There exists a canonical extension of the action of* Γ *on* $\partial\Gamma$ *to an action of* G *on* $\partial\Gamma$. *This action has the property that it is* G*-equivariant with respect to* π_d.

Proof. Every g in G defines a partial action on by conjugation with domain $\Gamma_{g^{-1}} = g^{-1}\Gamma g \cap \Gamma$ and codomain $\Gamma_g = g\Gamma g^{-1} \cap \Gamma$. Since both subgroups $\Gamma_{g^{-1}}$ and Γ_g have finite index in Γ, their Gromov boundaries coincide with $\partial\Gamma$ ([14]) Moreover as the boundary is independent of the chosen system of generators (see e.g. [15]) and hence the proof in [8] extends to this situation to prove that the conjugation action by g gives a continuous automorphism of $\partial\Gamma$ that clearly extends the action of Γ.

Recall that in [33], page 781 (see also [34]) the projection π_d is constructed as follows. One fixes a point $z \in \mathbb{H}$ and if $\zeta = w_1 w_2 ...$ is the infinite word expression in terms of the generators then

$$\pi_d(\zeta) = \lim_{n\to\infty} w_1 w_2 ... w_n(z), \tag{9}$$

The above definition is independent of z ([34]). By [24] the conjugacy action of G on wors in Γ is that of a Turing machine, which by the arguments in cooper eventually preserves the init in the iterated transformation of the init words in ζ. By combining this with the formula 9 and because that formula doesn't depend on the choice of z it follows that π_d is G-equivariant.

□

Remark 8. A possible way to describe the action of $g \in G$ on $\partial\Gamma$ is to consider along $\zeta \in \partial\Gamma$ a tube T of fixed width (greater than the lenght of coset representatives for $\Gamma_{g^{-1}}$. Then $T \cap \Gamma_{g^{-1}}$ is infinite, and gTg^{-1} will be contained in an infinite tube, of eventually larger width, that unequivocally defines the action of g on ζ.

Lemma 9. *The canonical embedding of $C_0(\partial\Gamma)$ inside $\ell^\infty(\Gamma)/c_0(\Gamma)$ has the property that the commutative C* algebra A generated by $C_0(\partial\Gamma)$ and*

$C(K)$ inside $\ell^\infty(\Gamma)/c_0(\Gamma)$ is isomorphic to $C_0(\partial\Gamma \times K)$. Here $C(K)$ is realized as the algebra generated by characteristic functions of cosets of subgroups of the form Γ_g, $g \in G$.

Moreover the partial action of G by conjugation on $\ell^\infty(\Gamma)/c_0(\Gamma)$ leaves A -invariant and on the factor $C_0(\partial\Gamma)$ induces the above action, while on $C(K)$ it induces the partial action.

Proof. One considers the commutative C*- algebra B in $\ell^\infty(\Gamma)/c_0(\Gamma)$ generated by characteristic functions of sets of the form:

(10) $$A_w = \{w_1 \in \Gamma \mid w_1 \text{ starts with } w\},\ w \in \Gamma.$$

Then A is the commutative C*-algebra generated by B and $C(K)$ in $\ell^\infty(\Gamma)/c_0(\Gamma)$. To prove that the algebra A is a faithfull realization of the commutative C*algebra $C(\partial\Gamma \times K)$, we need to exhibit a state (measure) on $\ell^\infty(\Gamma)/c_0(\Gamma)$ whose pushback to A is a product state. To do this let ω be a free ultrafilter on $\mathcal{N}$. For $n \in \mathbb{N}$ Let B_n be the ball of radius n in Γ and let μ_ω be the Loeb ([20], [19]) counting measure associated to the sets $(B_n)_{n\in\mathbb{N}}$ and the ultrafilter ω There is an obvious pull back from the Loeb space associated to the counting measure to the Stone-Cech compactification $\beta\Gamma$ and thus a further pushback of the Loeb measure to $\partial\Gamma \times K$. It is tautological that the marginal of the pushback measure on $\partial\Gamma$ is the homogenous measure (with respect to the generators use the construct the balls $B_n, n \in \mathbb{N}$).

The results in [6] prove that for a coset $C \subseteq K$ of a group of the form K_g, $g \in G$, we have that

$$\lim_{n\to\infty} \frac{\text{card }(B_n \cap C)}{\text{card }(B_n)} = \frac{1}{[\Gamma : \Gamma_g]}.$$

Hence the marginal of the pushback of the Loeb measure on K is the invariant Haar measure. Thus the algebra A is the algebra $C(\partial\Gamma \times K)$.

For $g \in G$, the continuity of the homeomorphism g on $\partial\Gamma$ shows that g will map (modulo $c_0(\Gamma)$ sets of the form $A_w \cap \Gamma_g$ into a reunion of such sets, and hence the conjugation action by g on $\ell^\infty(\Gamma)/c_0(\Gamma)$ invariates the algebra A. By the previous Remark, the action has the form described in the statement on the two factors.

□

In analogy with formula 10, for $w \in \Gamma$, let

$$A^w = \{w_1 \in \Gamma \mid w_1 \text{ ends with } w\},. \tag{11}$$

Theorem 10. *Let $G \times G^{\mathrm{op}}$ act on $\partial\Gamma \times K \times \partial\Gamma^{\mathrm{op}}$ by letting G act on $\partial\Gamma$ as in Lemma 7, and trivially on $\partial\Gamma^{\mathrm{op}}$. Similarly let G^{op} act on $\partial\Gamma^{\mathrm{op}}$ as in Lemma 7, and trivially on $\partial\Gamma$. We let $G \times G^{\mathrm{op}}$ act by partial isomorphisms on K. Let C be the commutative C*subalgebra of $\ell^\infty(\Gamma)/c_0(\Gamma)$ generated by characteristic functions of the form A_{w_1}, A^{w_2}, $w_1, w_2 \in \Gamma$ and characteristic functions of cosets in Γ of subgroups of the form Γ_g, $g \in G$.*

There C is isomorphic to $C(\partial\Gamma \times K \times \partial\Gamma^{\mathrm{op}})$ and the partial action of $G \times G^{\mathrm{op}}$ on $\ell^\infty(\Gamma)/c_0(\Gamma)$ invariates C and induces the action described above.

Proof. The fact that the algebra C is isomorphic to $C(\partial\Gamma \times K \times \partial\Gamma^{\mathrm{op}})$ is proved by the same argument based on Loeb measures as in the proof of the previous lemma.

Because Γ^{op} acts trivially on the algebra generated by the characteristic functions as in formula 10 and Γ acts trivially on characteristic functions as in formula 11 it follows that the action of $G \times G^{\mathrm{op}}$ on $\ell^\infty(\Gamma)/c_0(\Gamma)$ splits as it asserted in the statement. □

Using the natural action of G on $\partial\Gamma$ that we constructed above we deduce from Corollary 6 the following

Theorem 11. *The crossed product algebra*

$$\mathcal{D} = C^*((G \times G^{\mathrm{op}}) \ltimes_{\epsilon,\epsilon} C_0(\partial\Gamma \times K \times \partial\Gamma^{\mathrm{op}}))$$

coincides with the reduced crossed product algebra

$$\mathcal{D}_{\mathrm{red}} = C^*_{\mathrm{red}}((G \times G^{\mathrm{op}}) \ltimes_{\epsilon,\epsilon} C_0(\partial\Gamma \times K \times \partial\Gamma^{\mathrm{op}})).$$

Proof. We use the fact that the projection $\pi_d : \partial\Gamma \to P^1(\mathbb{R})$ ([33]) is bijective except of the rational points where it is 2 to 1. We use Takesaki's disintegration ([36], Chapter X, Theorem 3.8) for the crossed product algebra. Hence in any representation on a Hilbert space H of the algebra $\mathcal{D}$, there exist a measure $\mu_{\mathcal{D}}$ on $\partial\Gamma \times K \times \partial\Gamma^{\mathrm{op}}$ such that the Hilbert space H becomes a field of Hilbert spaces over $L^\infty(\mathcal{D}, \mu_{\mathcal{D}})$. If $\mu_{\mathcal{D}}$ gives mass 0 to the rational points then the result is exactly the content of Theorem 2. Otherwise we use the result in theorem 2 for a double copy of the algebra in 2, applied to the case when the measure is supported on the rational points. Every measure is the sum of two singular measures verifying these conditions, and the crossed product splits as a direct sum.

□

We can now conclude the proof of Theorem 1

Proof. Theorem 1

Indeed because of Theorem 10 we can extend the representation Π_Q of the algebra $\mathcal{A}$ to a representation $\widetilde{\Pi_Q}$ of the algebra $\mathcal{D} = C^*((G \times G^{\mathrm{op}}) \ltimes_{\epsilon,\epsilon} C_0(\partial\Gamma \times K \times \partial\Gamma^{\mathrm{op}}))$ into $Q(l^2(\Gamma))$. Because of Theorem 11 this factorizes to the reduced crossed product and hence also the restriction of $\widetilde{\Pi_Q}$ to A factorizes to the reduced crossed product algebra $\mathcal{A}_{\mathrm{red}}$.

$\square$

4. APPLICATION TO HECKE OPERATORS

Recall from [2] that the algebra $\mathcal{H}_0$ of linearly generated by the double cosets $[\Gamma\sigma\Gamma]$, for $\sigma \in G$ is represented into $B(l^2(\Gamma\backslash G))$, and correspondingly the reduced C*- Hecke algebra $\mathcal{H}_{\mathrm{red}}$ is the C*-algebra generated by the image of $\mathcal{H}_0$.

In [30] (see also [29]) we proved that if π_{13} is the 13-th projective unitary representation in the analytic series of representations of G_r and ζ is a cyclic trace vector (see [17], [18]) then the following sum converges

$$t([\Gamma\sigma\Gamma]) = \sum_{\theta\in\Gamma\sigma\Gamma} < \pi_(13)(\theta)\zeta, \zeta > \theta$$

and defines an element in ${C^*}_{\mathrm{red}}(G, \epsilon)$. Moreover it is proven in [30] (see also [29]) that

Theorem 12. *(1)The application mapping $[\Gamma\sigma\Gamma]$ into $t([\Gamma\sigma\Gamma])$, $\sigma \in G$ extends by linearity to a trace preserving isomorphism from $\mathcal{H}_{\mathrm{red}}$ into ${C^*}_{\mathrm{red}}(G, \epsilon)$.*

(2)The map defined by linear extension by the formula

$$\Psi([\Gamma\sigma\Gamma]) = E^{\mathcal{L}(G,\epsilon)}_{\mathcal{L}(\Gamma,\epsilon)}(t([\Gamma\sigma\Gamma]) \cdot t([\Gamma\sigma\Gamma])), \sigma \in G$$

is a $*$-homeomorphism fro $\mathcal{H}_0$ into $\mathcal{L}(\Gamma, \epsilon)$.

(3)If we extend $\Psi([\Gamma\sigma\Gamma])$ to $l^2(\Gamma)$ which is identified to the L^2-space associated to the II_1 factor $\mathcal{L}(\Gamma, \epsilon)$, then $\Psi([\Gamma\sigma\Gamma])$ viewed as an element $\widetilde{\Psi([\Gamma\sigma\Gamma])}$ in $B(l^2(\Gamma))$ is unitarily equivalent to the classical Hecke operator ([12]) associated to $[\Gamma\sigma\Gamma]$ acting on Maass forms on the upper halfplane.

(4) The linear application mapping $[\Gamma\sigma\Gamma]$ into

$$S([\Gamma\sigma\Gamma]) = \chi_K(t([\Gamma\sigma\Gamma]) \otimes t([\Gamma\sigma\Gamma]))\chi_K$$

takes values into the C-algebra $\mathcal{A} = C^*((G\times G^{\rm op})\ltimes_{\epsilon,\epsilon} C(K))$, and it extends to a trace preserving isomorphism from $\mathcal{H}_{\rm red}$ into $\mathcal{A}$.*

As a corollary of the above theorem from [30] (see also [29]) and because of Theorem 1 we obtain

Corollary 13. *The linear application obtained by linear extension by mapping $[\Gamma\sigma\Gamma]$ into the projection $\pi_Q(\Psi(\widetilde{[\Gamma\sigma\Gamma]}))$ into the Calkin algebra $Q(l^2(\Gamma))$, extends to a C*-isomorphism from $\mathcal{H}_{\rm red}$ with values into $Q(l^2(\Gamma))$*

Proof. Because of Theorem 1 this will certainly hold true if we prove that in the Calkin algebra we have the equality:

$$\Pi_Q(S([\Gamma\sigma\Gamma])) = \pi_Q(\Psi(\widetilde{[\Gamma\sigma\Gamma]})), \sigma \in G.$$

Recall that π_Q is the projection from $B(l^2(\Gamma))$ into the Calkin algebra, while Π_Q is the representation of the algebra $\mathcal{A}$ obtained by composing the representation of $\mathcal{A}$ into $B(l^2(\Gamma))$ with the projection into the Calkin algebra.

But in the definition of $\Psi(\widetilde{[\Gamma\sigma\Gamma]})$ in point (3) of Theorem 12, extending $\Psi([\Gamma\sigma\Gamma])$ as a completely positive map on the von Neumann II_1 factor to an operator acting on $l^2(\Gamma)$ amounts to transform the conditional expectation $E^{\mathcal{L}(G,\epsilon)}_{\mathcal{L}(\Gamma,\epsilon)}$ in point (2) of the above theorem into left and right multiplication by the characteristic function of Γ. This proves that the image of $S([\Gamma\sigma\Gamma]) \in \mathcal{A}$ in the representation of the algebra $\mathcal{A}$ into $B(l^2(\Gamma))$ coincides with $\Psi(\widetilde{[\Gamma\sigma\Gamma]})$ for $\sigma \in G$. Hence projecting onto the Calkin algebra their images coincide ☐

As a corollary we obtain straightforward from the previous corollary that

Corollary 14. *For $\sigma \in G$, the norm, in the Calkin algebra, of the Hecke operator $\Psi(\widetilde{[\Gamma\sigma\Gamma]})$ is equal to to the norm of the Hecke operator associated* *to* $[\Gamma\sigma\Gamma]$ *in the reduced Hecke algebra* $\mathcal{H}_{\rm red}$ *(i.e. the norm of* $[\Gamma\sigma\Gamma]$ *viewed as an operator acting on* $l^2(\Gamma\backslash G)$*).*

Let T_{p^n} be the Hecke operator, acting on Maass forms, associated to $\sigma_{p^n} = \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}$, n a positive integer. As proved in part (3) in Theorem 12, this is unitarily equivalent to $\Psi(\widetilde{[\Gamma\sigma_{p^n}\Gamma]})$. Hence we obtain

Theorem 15. *The spectrum of the Hecke operator* T_p*, acting on Maass forms, in the Calkin algebra (the essential spectrum) is contained in the interval* $[-2\sqrt{p}, 2\sqrt{p}]$*. Consequently, any given open interval containing* $[-2\sqrt{p}, 2\sqrt{p}]$*, contains at most a finite number of eigenvalues of* T_p *that lie outside the interval* $[-2\sqrt{p}, 2\sqrt{p}]$*.*

Proof. By part (3) of Theorem 12 the essential spectrum of T_p coincides with the essential spectrum of $\Psi(\widetilde{[\Gamma\sigma_p\Gamma]})$. By Corollary 14, the spectrum, in the Calkin algebra of $\Psi(\widetilde{[\Gamma\sigma_p\Gamma]})$ coincides with the spectrum of the double coset $[\Gamma\sigma_p\Gamma]$ viewed as an operator acting on acting on $l^2(\Gamma\backslash G)$. But this is equal to the norm of the radial element χ_1 (sum of words of lenght 1) in the free group $F_{(p+1)/2}$. This is because the $\mathcal{H}_{\rm red}$ is identified ([16]) with the radial algebra ([28]) in the free group. By [7] the spectrum of χ_1, viewed as, an element of ${C^*}_{\rm red}(F_{(p+1)/2})$ is the interval $[-2\sqrt{p}, 2\sqrt{p}]$ □

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