E1

A generalization of a theorem by Ihara and ~~its relation to Akemann~~ a local Akemann Ostrand phenomena.

Abstract. In this paper we give a generalization of Ihara's theorem for every prime number $p \geq 2$ the group

asserting that $G = PSL_2(\mathbb{Z}[\frac{1}{p}])$ $= \bar{G}^{re} \times \bar{G}^{p} =$

that is embedded into $PSL_2(\mathbb{R}) \times PGL_2(\mathbb{Q}_p)$,

by proving that $G \times G^{op}$ acts properly discontinuous, by left and right multiplication on $\underbrace{\bar{G}^{re} \times (\bar{G}^{re})^{op} \times (K \backslash \Gamma)}_{\text{the direct product}}$,

where $\Gamma = PSL_2(\mathbb{Z})$ and K is the standard profinite completion of K.

Using Zimmer amenability and Löeb type construction we deduce the the

groupoid C^*-algebra, of the groupoid $\mathcal{G} = (G \times G^{op}) \ltimes K$ and edges $G \times G^{op}$ with base space K, (where $G \times G^{op}$ acts by partial transformations), has the property that ~~the~~ its canonical embedding of $\ell^\infty \Gamma$ into the Calkin-algebra $Q(\ell^2 \Gamma)$ has the property that factorizes to the reduced C^*-algebra of the groupoid ~~K~~. $\mathcal{G}$ (restricted to $G \times G$). This is a local version of well known result first recognized by Akemann Ostrand.

§3

Definitions and outline of the proof.

In this paper p is a fixed number $p \neq 2$, Γ is $PSL_2(\mathbb{Z})$, $G = PGL_2(\mathbb{Z}[\frac{1}{p}])$ which is a dense subgroup of $\overline{G}^{\infty} = PSL_2(\mathbb{R})$ and of $\overline{G}^{p} = PGL_2(\mathbb{Q}_p)$. We let $\sigma_{p^n} = \begin{pmatrix} 1 & 0 \\ 0 & p^n \end{pmatrix}$, $n \geq 1$ an integer and let Γ_{p^n} be the finite index subgroups

$$\Gamma_{p^n} = \Gamma \cap \sigma_{p^n} \Gamma (\sigma_{p^n})^{-1}$$

It is easy to see that $\Gamma_{p^n} \supseteq \Gamma^{0}_{p^n}$, the minimal normal subgroups, still of finite index the kernel of Γ into $PSL_2(\mathbb{Z}_{p^n})$.

E4

Let K be the profinite completion of Γ with respect to the family $(\Gamma^{0}_{p^n})_{n\geq 1}$. Then $K = PSL_2(\mathbb{Z}_p)$, where $\mathbb{Z}_p$ are the p-adic integers.

It was proven by Ihara ([Ih]), that with respect to the canonical, diagonal embedding $G \subseteq \overline{G}^{ve} \times \overline{G}^{p}$, the group G, although being dense in each of the two factors, it is discrete (a lattice) in the product $\overline{G}^{ve} \times \overline{G}^{p}$.

We prove in this paper that one possible generalization is the following.

<u>Theorem</u> The action of ~~$G\times G$~~ the groupoid $\mathcal{G} = (G \times G^{op}) \rtimes K$ on $\overline{G}^{ve} \times (\overline{G}^{ve})^{op} \times K$, where the action on $\overline{G}^{ve} \times (\overline{G}^{ve})^{op}$ is by left multiplication, and on K by left and right

ε1

right multiplication ~~is~~ properly discontinuous

As a ~~xxx~~ Corollary, using lemma 0 in [Kosmal

we obtain, using Zimmer definition of
~~Gn~~

amenable action (via equivariant conditional

expectation)

<u>Corollary</u> The action of ~~is~~ the

groupoid $\mathcal{G}$ on $\tilde{G}^{re} \times (\tilde{G}^{re})^{op} \times (K \setminus \Gamma)$

is amenable / [~~?~~ Claire Renault] ?

We will use this to prove that

the ~~xxx~~ ~~repr~~ canonical representation of

$C^*(\mathcal{G})$ into $Q(\ell^2(\Gamma))$ ~~is~~ factorizes to

$C^*_{red}(\mathcal{G})$.

The representation of $C^*(\mathcal{G})$ into $Q(\ell^2(\Gamma))$ is constructed as follows. We let $\mathcal{G}$ act on Γ as a ~~partial~~ a groupoid by partial transformation. There is a canonical action of $C(K) \subseteq \ell^\infty(\Gamma)$ into $\ell^\infty(\Gamma)/c_0(\Gamma)$ and hence this gives a representation of $\pi: C^*(\mathcal{G}) \to B(\ell^2(\Gamma))$

$$\to B(\ell^2(\Gamma))/K(\ell^2(\Gamma))$$

The main theorem of this paper is that

<u>Theorem</u> The representation $\pi: C^*(\mathcal{G})$ into $Q(\ell^2(\Gamma))$ factorises to $C^*_{red}(\mathcal{G})$

To analyze the states corresponding on $C^*(\gamma)$ to this embedding (π), we use the Calkin construction. It is sufficient to consider a non-trivial ultrafilter ω and consider the states of the form $\omega_{\xi,\eta}\circ\pi$, where $\omega_{\xi,\eta}$ is the functional on $B(\ell^2(\Gamma))$ of the form

$$X \to \lim_{n\to\omega} \langle X\xi_n, \eta_n\rangle$$

and ξ_n, η_n weakly convergent to zero.

It is then a simple fact that we may reduce to $\xi_n = \frac{1}{\sqrt{A_n}}\chi_{A_n}$, where A_n are subsets of Γ, avoiding eventually any given subset.

E8

We consider the canonical boundary of Γ, and consider it from the left and from the right. It corresponds to considering the compactification of Γ by sets of words, starting, or ending with a given word. We denote the corresponding boundaries with $\partial_r\Gamma$ and $\partial_\ell\Gamma$

We consequently have an embedding

$$C^*(\mathcal{G}) \subseteq C^*(\mathcal{G} \times (\partial_r\Gamma \times \partial_\ell\Gamma \times K)) \subseteq Q(\ell^2(\Gamma))$$

The states $\left(\frac{1}{\sqrt{\operatorname{card} A_n}}\chi_{A_n}\right)_n$ gives rise to a Loeb probability measure that we denote by η_{ω, A_n} on a Loeb probability space that we denote by $\mathcal{C}_\omega((A_n))$. There is a canonical projection $p_\omega : \mathcal{C}_\omega((A_n)) \to \partial_e \Gamma \times \partial_v \Gamma \times K$, simply by considering limit after ω.

We consider the pushback $p_\omega^*(\eta_\omega)$ on $\partial_e \Gamma \times \partial_v \Gamma \times K$, and use Ihere lemma to conclude that the part of $p_\omega^*(\eta_\omega)$ that gives no mass to $\Gamma \subseteq K$, gives rise to a state that factorizes to $C^*_{red}(G)$.

The remaining part is similar to the action of G by conjugation on states and we use exactness of G.

A Generalization of a theorem by Ihara

In this section we consider the following generalization of Ihara's theorem.

We recall the notations from introduction

G is $PGL_2(\mathbb{Z}[\frac{1}{p}])$. We consider its closure $\overline{G}^{re} = PSL_2(\mathbb{R})$ and $\overline{G}^{p} = PGL_2(\mathbb{Q}_p)$. It is ~~well known that Ihara's~~ Let K be the profinite completion of Γ with respect to the groups $(\Gamma^0_{p^n})_{n \geq 1}$ ~~[illegible]~~. ~~The Ihara's theorem asserts that~~ We consider the groupoid action of $G \times G^p$ on K, by left right multiplication. Thus (g_1, g_2) will have as domain $g_1^{-1} K g_2 \cap K$ and will map this into $K \cap g_1 K g_2^{-1}$

E11

We denote this groupoid by $\mathcal{G}$.

It clearly acts on $\mathcal{Y} = \bar{G}^{r} \times \bar{G}^{re} \times K$,

the action beeing given by

$$(g_1, g_2)[h_1, h_2, k] = (g_1 h_1, g_2 h_2, g_1 k g_2)$$

if $h_1, h_2 \in \bar{G}^{re}$, k is in the domain of (γ_1, γ_2)

Ihara's theorem asserts that G is a lattice in $\bar{G}^{re} \times \bar{G}^{p}$. The essence of the theorem is that the two toppologies are very different, and a sequence converging in the topology of real manifolds cannot be convergent in the p-adic topology.

In particular $(\bar{G}^{re} \times \bar{G}^{p})/G$ is a nice space isomorphic to fundamental domain.

We extend the interplay of the two topologies by proving that ~~G~~ $\mathcal{G}$ on ~~$\mathcal{Y}$ acts properly~~ ~~discontinuous~~ $\mathcal{Y} = \bar{G}^{re} \times (\bar{G}^{re})^{p} \times (K \setminus \Gamma)$

acts properly discontinuous, and consequently the quotient is still Hausdorff and hence by a theorem of Koszul ([Ko]) it still has a fundamental domain (as a groupoid).

Note that this means that the space of orbits of Y/G is separated. Thus necessary we have to exclude $\bar{G}^{-ve} \times G^{-ve} \times \Gamma$, as $\{(g_1, \gamma^{-1} g_2 \gamma^{*})\}_{\gamma \in G}$ which acts on the ~~...~~ fiber over γ in $\bar{G}^{-ve} \times G^{-ve} \times \{\gamma\}$ has as closure all of $\bar{G}^{-ve} \times G^{-ve} \times \{\gamma\}$

<u>Theorem</u> ~~Proof~~ The quotient space $\mathcal{Y}/\mathcal{G}$ is Hausdorff.

<u>Proof</u>. In the sense mentioned in the paper [Kapilovich, A note on properly discontinuous actions], we have to prove that there are not $\underline{g}^n = (g_1^n, g_2^n)$ in $G \times G^P$ tending to infinity and $\alpha = (g_1^0, g_2^0, h)$, $\beta = (g_1', g_2', \ell)$ in $\mathcal{Y}$ such that, ~~$g^n x$ converges to~~ and $\alpha^n \to \alpha$, such that $\underline{g}^n \alpha_n \to \beta$.

(The points α, β are called in [Kap], dynamically correlated).

E15

Assume that $\alpha^n = (g_1^n, g_2^n, h_n)$

Then the above conditions imply that

$$g_j^n g_j^0 \text{ converges to } g_j^1, \; j = 1, 2$$

and h_n is in the domain of (g_1^n, g_2^n) and

$$g_1^n h_n g_2^n \text{ converges to } l \in K$$

It then follows that (g_1^n, g_2^n) converges to an element (g_1, g_2) which necessary belongs to $[\bar{G}^{ve} \times (\bar{G}^{ve})^{op}] \setminus (G \times G^{op})$, since (g_1^n, g_2^n) converges to infinity and that h_n belongs to the domain of (g_1^n, g_2^n) with $(g_1^n, g_2^n) \cdot (h_n) = g_1^n h (g_2^n)^{-1}$ converging to l. while h_n is converging to h

E16

We have that k, l are not in Γ

Otherwise ~~since~~ for example if $k \in \Gamma$, we could take $= \gamma$

$g_2^n = \gamma^{-1} g_1^n \gamma$ and then $g_1^n \gamma g_2^n$

$= g_1^n \gamma \gamma^{-1} (g_1^n)^{-1} \gamma = \gamma$, but g_1^n, could converge

easily to some $g_1 \in \overline{G}^{\kappa} \setminus G$ and likewise

will g_2^n converge to $\gamma^{-1} g_1 \gamma$.

- Then k is represented by ~~[illegible]~~ an

intersection $\zeta_n \Gamma_{g_1^n}$ and necessary then

g_2^n would be of the form $\zeta_n^{-1} g_1^n r_n$, where

$r_n \; \Gamma (g_1^n)^{-1}$ would converge to l.

But ζ_n^{-1}, r_n converge to ~~to~~ infinite makes ~~[illegible]~~ p-adic

numbers (i.e. non integers) and they are separable (i.e. no simplifications in the matrix product)

from the left and right by g_n, and hence

g_2^n cannot converge in the topology of $PSL_2(\mathbb{R})$.

t Σ 17

Essentially we have that $(g_1^n, g_2^n) \xrightarrow{k_n} (g_1, g_2)$ and $(g_1^n, g_2^n) \to e$,

while $h_n \to h$

Now $g_2^n = r_n g_1^n s_n$ automatically, where r_n are coset reps of $\Gamma_{(g_1^n)^{-1}}$, s_n are coset reps of $\Gamma_{(g_1^n)}$, and this writing is unique up to $\Gamma_{(g_1^n)^{-1}}$ at left which jumps to the right in $\Gamma_{(g_1^n)}$

In any case $\Gamma_{(g_1^n)} r_n^{-1}$ converges to the (h) (i.e. $(\exists)\ h_n \in \Gamma_{(g_1^n)} r_n^{-1}$, ~~with~~ with $h_n \to h$ and $(\Gamma_{g_1^n}) s_n$ converges to e. Since both $h, e \in K \setminus \Gamma$, r_n, s_n are p-adic integers, which viewed in real topology would converge to ∞

te18

Assume $g_1^n = \begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then I'll have with $r_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$

$s_n = \begin{pmatrix} x_n & y_n \\ z_n & t_n \end{pmatrix}$ that

$$a_n (p_1^n q_1^n) + b_n (p_2^n q_1^n) + c_n (p_1^n q_2^n) + d_n (p_2^n q_2^n)$$

converges in $\mathbb{C}$.

where (p_1^n, p_2^n) could be either (α_n, β_n) or (γ_n, δ_n)

and (q_1^n, q_2^n) could be either (x_n, z_n) or (y_n, t_n)

Now to converge in $\mathbb{C}$ means that

the points $(p_1^n q_1^n, p_2^n q_1^n, p_1^n q_2^n, p_2^n q_2^n)$

stay in a "horizontal" finite band along

kernel of $(x_1, x_2, x_3, x_4) \to a x_1 + b x_2 + c x_3 + d x_4$

$t \in 15$

On the other hand the set of points on which we are evaluating is of the form $(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)}) \in \mathbb{Z}^4$

with $\frac{x_1}{x_2} = \frac{x_3}{x_4}$, but more than that

$\frac{x_1^{k_1}}{x_2^{k_1}} = \frac{x_3^{k_2}}{x_4^{k_2}}$ if $k_{1,2} = \{1,2\}$ or $k_{1,2} = \{3,4\}$

(and a symmetric relation). In addition there is a hidden condition ~~on~~ which says that determinants are equal to 1

The shape of $\{(x_1^k, x_2^k, x_3^k, x_4^k) \mid x_i^k \in \mathbb{R}\}$ is so that it can not be contained at ∞ in any product of four balls.

Σ 20

By using the lemma 5 in Kostant lecture [Ko, Tate institute, group transformations)

we obtain

$:= (G \times G^{op}) \times K$

Corollary The action of $\mathcal{G}$ on $\mathcal{Y} = G^{-ve} \times G^{-ve} \times (0,1$

has a fundamental domain

Proof. The fact that $\mathcal{G}$ is a groupoid is irrelevant for the proof, and obviously $\mathcal{Y}$ is paracompact.

E21

Amenability of the action of $\mathcal{G} = (G \times G^{\varphi}) \ltimes K$ on $\partial_e\Gamma \times \partial_v\Gamma \times (K \setminus \Gamma)$

In this section we use the result in the previous section to prove the amenability.

Because of the result in the previous section we will prove first the amenability of the action of $\mathcal{G} = (G \times G^{\varphi}) \ltimes K$ on $\overline{G}^{ve} \times (\overline{G}^{ve})^{\varphi} \times (K \setminus \Gamma)$. Then we will use the fact that $\partial_v\Gamma$ is essentially the quotient by an amenable subgroup of $\overline{G}^{ve}$, and use results in [Monin, Monolli], [Kaimanovich, Haja

E22

We first prove

<u>Lemma</u> $\mathcal{G}$ acts amenably on $\bar{G}^{-ve} \times (G^{-ve})^p \times (K \backslash \Gamma)$

(amenability is in the sense of [Ah Ren]).

<u>Proof</u> We will use Zimmer [Zi] original definitions, and use the equivalences established in this paper (see also [Petterson, preprints)

We have consequently have to construct a

a $\mathcal{G}$-equivariant conditional expectation

$$\Phi : L^\infty(\mathcal{G} \times \mathcal{Y}, \nu) \to L^\infty(\mathcal{Y}, \nu)$$

for any quasi-equivariant measure ν on

$$\mathcal{Y} = \bar{G}^{-ve} \times (G^{-ve})^p \times (K \backslash \Gamma)$$

Because of the existence of the fundamental domain

$\underline{E23}$

We use the equivalent Gesuir completely positive expectati (see [Zimmer], also e.g. Petkov)

$\underline{\text{Gesuir}}$ The fundamental domain gives an embedding of $\Phi : \ell^\infty(G) \longrightarrow \ell^\infty(\bar{G}^{ve} \times \bar{G}^{ve} \times (K \setminus \Gamma))$

The measure on product of $\bar{G}^{ve} \times \bar{G}^{ve} \times (K\setminus\Gamma)$ is standard Haar measure

~~This is because of it~~

Here we look at G rather as a countable group of partial transformation and prove equivariance under G.

On the other hand $\bar{G}^{ve} \times (\bar{G}^{ve})^{op}$ acts amenably on $\bar{G}^{ve}/P \times (\bar{G}^{ve}/P)^{op}$ if P is a closed amenable subgroup of $\bar{G}^{ve} = PSL_2(\mathbb{R})$

Hence there exists a $\bar{G}^{ve} \times \bar{G}^{ve}$ equivariant map

$$E : \ell^\infty([\bar{G}^{ve} \times \bar{G}^{ve}] \times \bar{G}^{ve}/P \times \bar{G}^{ve}/P)]$$

23

into $L^\infty(\bar{G}/P \times \bar{G}/P)$

We ~~dress compose~~ tensor this with the Id of $L^\infty(K\backslash\Gamma)$

Then $\Phi\left(\Xi \otimes Id_{L^\infty(K\backslash\Gamma)}\right) \circ \left(\Phi \otimes Id_{L^\infty(\bar{G}/P)\otimes L^\infty(\bar{G}/P) \otimes L^\infty(K\backslash\Gamma)}\right)$

is a $\mathcal{G}$ equivariant map (as $\mathcal{G}$ is a subgroup of $\bar{G}\times\bar{G}^{\text{ve}}$ if forget the domain)

is a $\mathcal{G}$ equivariant map from

$$L^\infty(\mathcal{G}) \otimes \left[L^\infty(\bar{G}/P) \otimes L^\infty(\bar{G}/P) \otimes L^\infty(K\backslash\Gamma)\right]$$

in $L^\infty(\bar{G}/P \times \bar{G}/P \times K\backslash\Gamma)$

Since this respect the domain of $\mathcal{G}$ when acting on $K\backslash\Gamma$

it follows that $\mathcal{G}$ acts amenably on $\bar{G}/P \times \bar{G}/P \times (K\backslash\Gamma)$. ∎

E20

§3. The boundary action of $G \times \bar{G}^{op}$.

In this section we describe the boundary action. ~~Recall~~ We consider the compactification $\bar{\Gamma}^{e}$ of Γ generated by ~~consisting of~~ the subsets of the form

A^1_w = {words starting with w} with respect canonical generators of $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$

A^2_w = {words ending with w}

$\Gamma_g = g\Gamma g^{-1} \cap \Gamma$ for $g \in G$

__Lemma.__ With this compactification the boundary $\partial_e \Gamma = \bar{\Gamma}^{e} \setminus \Gamma$ becomes $\partial_e \Gamma \times \partial_P \Gamma \times K$, where $\partial_P \Gamma$ is the ~~usual~~ standard Poisson boundary of Γ.

E27

<u>Proof.</u> It is sufficient to note that any intersection of such sets is infinite. Indeed we can have infinitely many words ~~as long~~ starting with w_1, ending with w_2 and belonging to a given coset of some Γ_j

Just take a ~~f~~ very long word starting with w_1, ending with w_2. Somewhere in the middle if word is long enough, we correct the word by one of the coset representatives so that it belongs to any specific coset of Γ_j in Γ.

E28

We describe the boundary action of $(G \times G^{op})$ on $\partial_e \Gamma \times \partial_v \Gamma \times K$.

Lemma. The action of $G \times G^{op} \times K$ is described as follows. ~~Th~~ G and G^{op} acts by a geodesic action, (described in Rainer as an automata) on $\partial_e \Gamma$ respectively $\partial_v \Gamma$, and (g_1, g_2) on K without preserving the domain.

Proof. By [Higson, + the guy in Hawaii], we know that $\Gamma \times \Gamma^{op}$ acts only ~~b~~ by the left component (respective by the right) component on $\partial_e \Gamma \times \partial_v \Gamma$.

E2#9

$= (g \cdot g^{-1})$

Eky σ_g if $g \in G \Gamma$, will act

by partial automorphisms. It will

only look at words that belong to Γ_g^{-1}

The fact that on infinite word σ_g is

well defined and yields another infinite word

follows from the paper [Grigorchuk, Kaimanovich, Nagnibeda].

(and implicitly the fact that the compactification is invariant by $G \times G$)

The action is described in the paper [Rhiman, automata Raimar]

In the end the action of (g_1, g_2) on

an infinite word, because also of the above

result by [Higson + ...] (see also [Ozawa],

[Brown, Ozawa], will depend only on g_1

on $\partial_e \Gamma$ and only on g_2 on $\partial_r \Gamma$.

The action of G on $\partial\Gamma$ is proved ([Spielberg], [Marin Marcolli]

to be a G equivariant covering (non bijective only

on a countable set of points) of G acting on $P_1(\mathbb{R})$

<u>Corollary</u>. The action of ~~G~~ the groupoid $\mathcal{G}$ on $\partial_v\Gamma \times \partial_e\Gamma \times (K \setminus \Gamma)$ is amenable.

<u>Proof</u>. This follows from the result in the previous chapter plus the fact mentioned above that action of G on $\partial_e\Gamma$ is one to one $\overset{G\text{ equivariant}}{\text{covers of}}$ $P^1(\mathbb{R}) = \overline{G}^{ve}/P$. (except for a countable set of points that have a double preimage. $\boxed{\not{}}$

Re

234

To construct the states corresponding to this representation we use the Calkin construction for a faithfull family of states on $Q(\ell^2(\Gamma))$. As proven by [Co], it is sufficient to consider all sequences

$\xi\ (\xi_n)$ of vectors in $\ell^2(\Gamma)$,

of unit vectors weakly converging to zero. Then one considers a free ultrafilter on $\mathbb{N}$.

let $\varphi_{\xi,\omega}(x) = \lim_{n\to\omega} (x\xi_n, \xi_n)$

Since ξ_n are vectors in $\ell^2(\Gamma)$, we may assume that ξ_n as functions on Γ are positive, with finite support, depending on n.

We are the interested in

$\varphi_{\xi,\omega} \circ \pi$ as states on $C^*(G \times_\epsilon (\Omega_\epsilon \Gamma \times \Omega_\epsilon K \times K))$

and these states are a faithfull family

E31

§ The groupoid C^*-algebra and its representation into the Calkin algebra.

We have an obvious representation, of the $C(\partial_e \Gamma)$ compactification of Γ in the previous representation into the algebra $\ell^\infty(\Gamma)/c_0(\Gamma)$ the Stone Cech compactification of ~~the~~ Γ.

This is ~~$G \times G$~~ $\mathcal{G}$-equivariant, where $G \times G^{op}$ is represented by left right multiplication operators, and $C(K)$ acts by multiplication operators ~~on~~ Γ the C^*-algebras

Thus we have a representation of $C^*(\mathcal{G}) =$

~~$\pi: C^*(\mathcal{G})$~~ $C^*(\mathcal{G} \ltimes (\partial_e \Gamma \times \partial_r \Gamma \times K))$

into $Q(\ell^2(\Gamma))$

~~Lemma~~

The states that have to be analysed come from $\underbrace{\varphi_{\xi,\omega} \circ \pi(gf)}_{\text{evaluating terms of the form}}$, where $gf \in \mathbb{C}(G \times \mathbb{C}[\mathbb{F}_\infty])$

$\varphi_{\xi,\omega}$ is not necessary a Koopman state

We note the following example.

Lemma

Let $A_n = \{\omega \mid |\omega| \leq n\}$, the set of words of lenght n. Consider $\xi_n = \frac{1}{\sqrt{\operatorname{card} A_n}} \chi_{A_n}$

Then the ~~Haa~~ Loeb measure $\nu_{\xi,\omega}$ has the property that the push backs on $\partial_2 \Gamma$ $(\Omega \partial_e \Gamma)$ is exactly the quasiinvariant measure considered in [Pi, Tdemonia] which gives a weight to A_w ~~equal to~~ that is inversely proportional to len

Then ~~$\theta_{\xi,\omega}$~~

Lemma. There exists probability measure space $(X_{\xi,\omega}, \eta_{\xi,\omega})$ and $\pi : X_{\xi,\omega} \Rightarrow \partial_\beta \Gamma \rightarrow \ldots$ such that

$\eta_{\xi,\omega}$ and an embedding ~~$C(\partial_e\Gamma \times \partial_r\Gamma)$~~

of $C(\partial_\beta \Gamma)$ (and thus of $C(\partial_e \Gamma \times \partial_r \Gamma \times K)$)

into $L^\infty(X_{\xi,\omega}, \eta_{\xi,\omega})$ such that the

state $\varphi_{\xi,\omega}^{\circ\pi}|_{C(\partial_\beta\Gamma)}$ is the pushback

of $\eta_{\xi,\omega}$

Proof. This is simply the Loeb measure construction. Note that composing with $g \in S$

$g(\pi^*(\eta_{\xi,\xi}))$ will be the measure on $C(\partial_\beta\Gamma)$

and $(\partial_r\Gamma \times \partial_e\Gamma \times K)$ obtained by composing with action of the transformation given by g

35

$\dagger$:

Observation The measure $\eta_{\xi,\omega}$ is not necessary quasi invariant, but one can still make ~~change~~ construct $g^*(\eta_{\xi,\omega})$ as G acts on $^*\Gamma$ (the non-standard) space,

One obtains that $\varphi_{\xi,\omega}(g) = \int \left(\frac{d g^*(\eta_{\xi,\omega})}{d\eta_{\xi,\omega}}\right)^{1/2}$

This might be explained better by saying that the minimal quasi invariant components occurring in $\eta_{\xi,\omega}$ depend on cardinality in $^*\mathbb{N}$ of supports of ξ.

Hence one may assume $\xi_n = \frac{1}{\sqrt{\operatorname{card} A_n}}$ const

and $\eta_{\xi,\omega}$ is Lebesgue counting measure, and the space of the quasiinvariant measure is $\bigcup g X_{\omega,\xi}$

Σ~~Q~~ 36

~~Phenomena~~

The Akemann Ostrand phenomena

First we assume that $\pi_K^*(\mu_{\xi,A})$.

where π_K : ~~$\partial_v\Gamma$~~ $X_\omega \Rightarrow K$, has the property that it gives value zero to Γ (is singular to counting measure on Γ)

Lemma. In this case $\varphi_{\xi,A}$ is equivalent a state on $C^*(G \times C(\partial_v\Gamma \times \partial_e\Gamma \times (K \cup \Gamma))$ and hence $\varphi_{\xi,A}$ is a state ~~$\not$~~ which factorizes to the $C^*_{red}(G \times C(\partial_v\Gamma \times \partial_e K \times K))$

Proof. Recall that we proved that $C^*(G \times C(\partial_v\Gamma \times \partial_e\Gamma \times K))$ is nuclear.

E37

Indeed for every quasiinvariant measure on $\mathcal{Y} = \partial_v \Gamma \times \partial_\ell \Gamma \times K$, the action of $\mathcal{G}$ is amenable and hence by lemma 3.3.7 in [Claire Renault] (see also Claire, TAMS paper, page 4158, paragraph 2) from fellow

it follows that $C^*(\mathcal{G} \times C(\partial_v \Gamma \times \partial_\ell \Gamma \times K(\Gamma)))$

is nuclear.

Thus $\varphi_{\xi,A}$ is a state continuous with respect topology $C^*_{red}(\mathcal{G} \times C(\mathcal{Y}))$ and hence with $C^*_{red}(\mathcal{G}, C^0(\mathcal{Y}, \nu))$

But in the GNS of $\varphi_{\xi,A}$, $C^*(\mathcal{G} \times C(\mathcal{Y}))$

is same as $C^*(\mathcal{G} \times C(\partial_\ell \Gamma \times \partial_v \Gamma \times K))$

and hence $\varphi_{\xi,\omega}$ factorizes to $C^*_{red}(\mathcal{G} \times C(\mathcal{Y}))$ $\square$

where ν is the minimal quasi invariant measure ~~containing~~ over $\pi_K^*(\eta_{\xi,A})$

In fact what we are doing here is we take $\nu =$ minimal quasi-invariant measure dominating $\nu_0 = \pi^*(\mu_{\xi,\omega}))$
$\partial_\Gamma \times \partial_\Gamma \times K \setminus \Gamma$

and observe that by nuclearity

$$C^*(\mathcal{G} \times L^\infty(\mathcal{Y}, \nu)) = C^*_{red}(\mathcal{G} \times L^\infty(\mathcal{Y}, \nu))$$
$(\partial_\Gamma \times \partial_\Gamma \times K)$

which is also equal to $C^*_{red}(\mathcal{G} \times L^\infty(\mathcal{Y}))$

Then $\varphi_{\xi,\omega} \circ \pi$ factorizes to a state on $C^*(\mathcal{G} \times L^\infty(\mathcal{Y}, \nu))$ and hence $\varphi_{\xi,\omega}$ factorizes to a state on $C^*_{red}(\mathcal{G} \times L^\infty(\partial_\Gamma \times \partial_\Gamma \times K))$.

$\boxtimes$

Note that $\pi(\mu_{\xi,\omega}))$ may not be quasi-invariant, but taking a larger ξ we expect it to become quasi-invari

This implies the minimal quasinvariant measure carrying ν_0 has infinite mass

~~Thm~~

~~Prop~~

~~Thy~~ ~~The Akemann Ostrand phenomena~~

~~holds true~~

Th. In general the state $\varphi_{\xi,\omega}\circ\pi$ factorizes to $C^*_{red}(G \times (\partial_h\Gamma \times \partial_e\Gamma \times K))$.

Proof. Assume that $\pi^*_K(\nu_{\xi,\omega}))$ has a singular part, i.e. has mass over Γ. Say this is $\sum_{\gamma\in\Gamma} \varepsilon_\gamma \delta_\gamma$

For simplicity assume that $\xi_n = \frac{1}{\text{conv}} \ldots$

E41

Then there exists for each δ, neighborhoods of δ, of the form $S^n \delta$

$$B(u,\delta)\left(S^{n}(\delta) \;\Gamma_{\sigma_\rho}\; \mathcal{R}(u,\delta) \right)$$

such that the mass of A_n inside this

i.e. $\dfrac{\operatorname{card} A_n \cap B(u,\delta)}{\operatorname{card} A_n}$ converges to ε_δ

By continuity we may assume only a finite number of them.

If we exclude from the Loeb space A_n, the intersections, we will get a Loeb space minus a direct reunion of Loeb spaces and this will be approximable with Loeb spaces to which the argument in the previous applies

$\Sigma 42$

Hence to finish the proof, we may assume by linearity that only one point in γ^Γ has nonzero mass

~~The Ridg~~

We analyze the state corresponding to ξ_0 X^0_ω corresponding the points giving mass to γ^e.

On an element $(g_1, s_2) \in \mathcal{G}$ the corresponding state φ_{ω,ξ_0} will be nonzero only if (g_1, s_2) stabilizes γ, i.e. if g_2 is of the form $\gamma^{-1} g_1^{-1} \gamma^e$

Hence by conjugating with γ^e, we may assume that $\gamma^e = e$, and the state φ_{ω,ξ_0} is non-vanishing only on $\{(s, s^{-1}) \mid s \in G\}$

But then, since the stabilizers of conjugation of G on Γ are all amenable,

$\Sigma 43$

The points A_n° will behave like they would be in G/H_0, where H_0 is an amenable group. The stab would be $C^*_{red}(\mathcal{G})$ and hence $C^*_{red}(\mathcal{G})$ $\square$