# UNIVERSITÀ DI ROMA TOR VERGATA

## Laurea Magistrale in Matematica Pura e Applicata

Corso di *EP-Elementi di Probabilità*
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**Exercise 1** Let $\xi$ be the solution (geometric Brownian motion) to the SDE

$$d\xi_t = b\xi_t\,dt + \sigma\xi_t\,dB_t$$

with the initial condition $\xi_0 = 1$.

a) Determine a real number $\alpha$ such that $(\xi_t^\alpha)_t$ is a martingale.

b) Let $\tau$ the exit time of $\xi$ out of the interval $]\frac{1}{2}, 2[$. Compute the probability $\mathrm{P}(\xi_\tau = 2)$.

**Exercise 2** Let $B$ be 1-dimensional Brownian motion and $\xi$ the Ornstein-Uhlenbeck process, solution of the SDE

$$\begin{aligned} d\xi_t &= b\xi_t\,dt + \sigma\,dB_t \\ \xi_0 &= x \end{aligned}$$

where $b, \sigma \in \mathbb{R}$. Let $\eta_t = \xi_t^2$.

Prove that $\eta$ is the solution of a SDE to be determined.

**Exercise 3** Let, in dimension 2, $L$ be the operator

$$L = \frac{1}{2}\,\frac{\partial^2}{\partial x^2} + x\,\frac{\partial}{\partial y} \tag{1}$$

and $\xi$ the diffusion having $L$ as its generator.

a) Compute the law of $\xi_t$ with the starting condition $\xi_0 = x$. Does it have a density with respect to the Lebesgue measure of $\mathbb{R}^2$? Is $L$ elliptic?

b) Answer the same questions as in a) for the operator

$$L_2 = \frac{1}{2}\,\frac{\partial^2}{\partial x^2} + y\,\frac{\partial}{\partial y}\,\cdot \tag{2}$$

**Exercise 4** Let $\xi^\varepsilon$ be the Ornstein-Uhlenbeck process solution to the stochastic differential equation

$$d\xi_t^\varepsilon = -\lambda\xi_t^\varepsilon\,dt + \varepsilon\sigma\,dB_t, \quad \xi_0^\varepsilon = x$$

where $\lambda \in \mathbb{R}$, $\sigma > 0$.

a) Prove that, for every $t \geq 0$

$$\xi_t^\varepsilon \underset{\varepsilon\to 0}{\overset{\mathcal{L}}{\to}} \mathrm{e}^{-\lambda t}x$$

b) Prove that the laws of the processes $\xi^\varepsilon$ (which, remember, are probabilities on the space $\mathscr{C} = C([0,T],\mathbb{R})$), converge in distribution to the Dirac mass concentrated on the path $x_0(t) = \mathrm{e}^{-\lambda t}x$ that is the solution of the ordinary equation

$$\dot{\xi}_t = -\lambda \xi_t, \quad \xi_0 = x \tag{3}$$

In other words the diffusion $\xi^\varepsilon$ can be seen as a small random perturbation of the ODE (3).

# Solutions

**Exercise 1.** a) Two possibilities: by Ito's formula, for every real number $\alpha$

$$d\xi_t^\alpha = \alpha\xi_t^{\alpha-1}\,d\xi_t + \frac{1}{2}\,\alpha(\alpha-1)\xi_t^{\alpha-2}\sigma^2\xi_t^2\,dt = \big(\alpha b + \frac{\sigma^2}{2}\,\alpha(\alpha-1)\big)\xi_t^\alpha\,dt + \sigma\xi_t^\alpha\,dB_t\ .$$

If the coefficient of $dt$ in the stochastic differential above vanishes, then $\xi^\alpha$ will be a local martingale and actually a martingale (why?). The condition is therefore $\alpha = 0$ (obviously) or

$$b + \frac{\sigma^2}{2}\,(\alpha - 1) = 0$$

i.e.

$$\alpha = 1 - \frac{2b}{\sigma^2}\ \cdot \tag{4}$$

Second possibility: we know that $\xi$ has the explicit form

$$\xi_t = \mathrm{e}^{(b-\frac{\sigma^2}{2})+\sigma B_t}$$

and therefore

$$\xi_t^\alpha = \mathrm{e}^{\alpha(b-\frac{\sigma^2}{2})+\alpha\sigma B_t}$$

which turns out to be an exponential martingale if

$$\alpha(b - \tfrac{\sigma^2}{2}) = -\alpha^2\frac{\sigma^2}{2}$$

from which we obtain again (4).

Remark however that the use of Ito's formula here requires an explanation, as the function $x \to x^\alpha$ is not defined on the whole of $\mathbb{R}$.

b) By the stopping theorem we have, for every $t \ge 0$,

$$\mathrm{E}[\xi_{\tau\wedge t}^\alpha] = 1\ .$$

As $t \to \xi_{\tau\wedge t}^\alpha$ remains bounded in $t$, we can apply Lebesgue's theorem and obtain

$$1 = \mathrm{E}[\xi_\tau^\alpha] = 2^\alpha \mathrm{P}(\xi_\tau = 2) + 2^{-\alpha}\big(1 - \mathrm{P}(\xi_\tau = 2)\big)$$

from which we find

$$\mathrm{P}(\xi_\tau = 2) = \frac{1 - 2^{-\alpha}}{2^\alpha - 2^{-\alpha}}\ \cdot$$

**Exercise 2.** By Ito's formula we have

$$d\eta_t = 2\xi_t\, d\xi_t + \sigma^2\, dt = (\sigma^2 + 2b\xi_t^2)\, dt + 2\sigma\xi_t\, dB_t$$

If we define

$$W_t = \int_0^t \Big(\frac{\xi_t}{\sqrt{\eta_t}}\, 1_{\{\eta_t \ne 0\}} + 1_{\{\eta_t = 0\}}\Big)\, dB_t \ . \tag{5}$$

then, by Corollary '`lem7.14`', $W$ is a real Brownian motion (the integrand is a process having modulus equal to 1) and

$$\sqrt{\eta_t}\, dW_t = \sqrt{\eta_t}\,\Big(\frac{\xi_t}{\sqrt{\eta_t}}\, 1_{\{\eta_t \ne 0\}} + 1_{\{\eta_t = 0\}}\Big)\, dB_t = \xi_t\, dB_t \ .$$

Hence $\eta$ solves the SDE

$$d\eta_t = (\sigma^2 + 2b\eta_t)\, dt + 2\sigma\sqrt{\eta_t}\, dW_t \tag{6}$$

of course with the initial condition $\eta_0 = x^2$.
• Remark that the coefficients of (6) do not satisfy Condition (A') (the diffusion coefficient is not Lipschitz continuous at 0). In particular in this exercise we prove that (6) has a solution but are unable to discuss uniqueness. a) The matrix of the second order coefficients is

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

whose square root, $\sigma$, is of course equal to $a$ itself. The corresponding SDE is therefore

$$d\xi_t = b\xi_t\, dt + \sigma\, dB_t$$

where

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and we know that this equation has the explicit solution

$$\xi_t = e^{bt}\xi_0 + e^{bt}\int_0^t e^{-bs}\sigma\, dB_s$$

and has a Gaussian law with mean $e^{bt}\xi_0$ and covariance matrix

$$\Gamma_t = \int_0^t e^{bs}\sigma\sigma^* e^{b^*s}\, ds \ .$$

We see that $b^2 = b^3 = \ldots = 0$. Hence

$$\mathrm{e}^{bu} = \sum_{k=0}^{\infty} \frac{b^k u^k}{k!} = I + bu = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$$

so that

$$\mathrm{e}^{bu}\sigma\sigma^*\mathrm{e}^{b^*u} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u \\ u & u^2 \end{pmatrix}$$

and

$$\Gamma_t = \begin{pmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & \frac{t^3}{3} \end{pmatrix} .$$

$\Gamma_t$ being invertible (determinant equal to $\frac{t^3}{12}$, the law of $\xi_t$ has a density.

b) The SDE now is of the same kind but with

$$b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{e}^{bu} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^u . \end{pmatrix}$$

Therefore

$$\mathrm{e}^{bu}\sigma\sigma^*\mathrm{e}^{b^*u} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^u \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\Gamma = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$$

which is not invertible so that there is no density with respect to Lebesgue measure.

**Exercise 4.** a) We know that the law of $\xi_t^\varepsilon$ is Gaussian with mean $\mathrm{e}^{-\lambda t}x$ and variance

$$\frac{\varepsilon^2\sigma^2}{2\lambda}\,(1 - \mathrm{e}^{-2\lambda t}) .$$

By Chebyshev inequality

$$\mathrm{P}(|\xi_t^\varepsilon - \mathrm{e}^{-\lambda t}x| \geq \delta) \leq \varepsilon^2\,\frac{\sigma^2}{2\lambda\delta^2}\,(1 - \mathrm{e}^{-2\lambda t}) \underset{\varepsilon\to 0}{\to} 0$$

so that $\xi_t^\varepsilon$ converges to $\mathrm{e}^{-\lambda t}x$ in probability and therefore in distribution.

b) It is sufficient to prove that

$$\mathrm{P}\Big(\sup_{t\leq T}|\xi_t^\varepsilon - \mathrm{e}^{-\lambda t}x| \geq \delta\Big) \underset{\varepsilon\to 0}{\to} 0 .$$

Actually this entails that the probability of $\xi^\varepsilon$ to be outside of a fixed neighborhood of the path $x_0(t) = \mathrm{e}^{-\lambda t}$ as goes to 0. Recall the explicit expression for $\xi^\varepsilon$:

$$\xi_t^\varepsilon = \mathrm{e}^{-\lambda t} x + \varepsilon\sigma \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda s}\, dB_s \ .$$

Therefore we are led to the following limit to be proved

$$\mathrm{P}\Big(\varepsilon \sup_{t\le T} \Big|\sigma \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda s}\, dB_s\Big| \ge \delta\Big) \underset{\varepsilon\to 0}{\to} 0 \ .$$

which is immediate, the process inside the absolute values being continuous and therefore the r.v.

$$\sup_{t\le T} \Big|\sigma \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda s}\, dB_s\Big|$$

is finite. Remark that, using the exponential inequality of martingales we might even give a speed of convergence.