# UNIVERSITÀ DI ROMA TOR VERGATA

## Laurea Magistrale in Matematica Pura e Applicata

Corso di *EP-Elementi di Probabilità*
P.Baldi
Tutorato 7 del 23 maggio 2016

**Exercise 1** Let, for $\lambda > 0$,

$$X_t = \int_0^t e^{-\lambda s} B_s \, ds$$

Does the limit

$$\lim_{t \to +\infty} X_t \tag{1}$$

exist? In which sense? Determine the limit and compute its distribution.

**Exercise 2** Let $\xi$ be the Ornstein-Uhlenbeck process that solves the SDE

$$d\xi_t = -\lambda \xi_t \, dt + \sigma \, dB_t, \quad \xi_0 = x$$

where $\lambda > 0$.

a) Show that $Y_t = e^{\lambda t} \xi_t$ is a martingale.

b) Show that $Z_t = e^{2\lambda t} \Big( \xi_t^2 - \frac{\sigma^2}{2\lambda} \Big)$ is a martingale.

c) Find numbers $\alpha, c$ such that $S_t = e^{\alpha t} (\xi_t^3 + c\xi_t)$ is a martingale.

**Exercise 3** Let $X$ be the solution of the SDE, for $t < 1$,

$$\begin{aligned} d\xi_t &= -\frac{1}{2} \, \frac{\xi_t}{1-t} \, dt + \sqrt{1-t} \, dB_t \\ \xi_0 &= x \end{aligned} \tag{2}$$

a) Find the solution of this equation and prove that it is a Gaussian process.

b) Compare the variance of $\xi_t$ with the corresponding variance of a Brownian bridge at time $t$. Is $\xi$ a Brownian bridge?

**Exercise 4** Let $B = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (B_t)_t, \mathrm{P})$ be a $m$-dimensional Brownian and $c, \alpha_1, \dots, \alpha_m \in \mathbb{R}$.

a) Given the martingale

$$Y_t = \sum_{i=1}^m \alpha_i B_i(t)$$

determine its associated increasing process.

b1) Given the process

$$X_t = \exp\Big(ct + \sum_{i=1}^{m} \alpha_i B_i(t)\Big)$$

Prove that there exist a real Brownian motion $(W_t)_t$ and coefficients $b, \sigma$, to be determined, such that $X$ is the solution of the SDE

$$dX_t = b(X_t)X_t\,dt + \sigma(X_t)X_t\,dW_t, \qquad X_0 = 1 \tag{3}$$

b2) Compute

$$\lim_{t\to+\infty} \mathrm{E}(X_t^2)$$

according to the values $c, \alpha_1, \ldots, \alpha_m$.

# Solutions

**Exercise 1.** We have, by integration by parts

$$\int_0^t \mathrm{e}^{-\lambda s} dB_s = \mathrm{e}^{-\lambda t} B_t + \lambda \int_0^t \mathrm{e}^{-\lambda t} B_t\, dt = \mathrm{e}^{-\lambda t} B_t + \lambda X_t$$

Now the stochastic integral at the right-hand side is a martingale bounded in $L^2$, therefore converging a.s. and in $L^2$. As $\lim_{t\to+\infty} \mathrm{e}^{-\lambda t} B_t = 0$ a.s. (recall the Iterated Logarithm law) and in $L^2$, we have that also the limit in (1) exists a.s. and in $L^2$, the limit being the r.v.

$$\frac{1}{\lambda}\int_0^{+\infty} \mathrm{e}^{-\lambda s} dB_s$$

which is a centered Gaussian r.v. with variance

$$\frac{1}{\lambda^2}\int_0^{+\infty} \mathrm{e}^{-2\lambda s}\, ds = \frac{1}{2\lambda^3}\,\cdot$$

**Exercise 2.** a) Ito's formula can here be applied in many ways. The simplest here is, possibly, to write

$$dY_t = \lambda \mathrm{e}^{\lambda t}\xi_t\, dt + \mathrm{e}^{\lambda t}\, d\xi_t = \mathrm{e}^{\lambda t}\big(\lambda\xi_t\, dt - \lambda\xi_t\, dt + \sigma\, dB_t\big) = \mathrm{e}^{\lambda t}\sigma\, dB_t\ .$$

b) Another way of applying Ito's formula is to recall that, for a diffusion process $\xi$ and a smooth function $u$, we have

$$du(\xi_t) = Lu(\xi_t)\, dt + u'(\xi_t)\sigma(\xi_t)\, dB_t\ .$$

Here

$$Lu(x) = -\lambda x u'(x) + \frac{\sigma^2}{2}\, u''(x)$$

and, for $u(x) = x^2 - \frac{\sigma^2}{2\lambda}$, we have $Lu = -2\lambda x^2 + \sigma^2$. Ito's formula then gives

$$\begin{aligned} dZ_t &= 2\lambda \mathrm{e}^{2\lambda t} u(\xi_t)\, dt + \mathrm{e}^{2\lambda t}\, du(\xi_t) = \\ &= 2\lambda \mathrm{e}^{2\lambda t} u(\xi_t)\, dt + \mathrm{e}^{2\lambda t} Lu(\xi_t)\, dt + u'(\xi_t)\sigma(\xi_t)\, dB_t = \\ &2\lambda \mathrm{e}^{2\lambda t}\Big(\xi_t^2 - \frac{\sigma^2}{2\lambda}\Big)\, dt + \mathrm{e}^{2\lambda t}\big(-2\lambda\xi_t^2 + \sigma^2\big) + 2\sigma\xi_t\, dB_t = \\ &= \mathrm{e}^{2\lambda t}\Big(2\lambda\xi_t^2 - 2\lambda\xi_t^2 - \sigma^2 + \sigma^2\Big)\, dt + 2\sigma\xi_t\, dB_t = \\ &= 2\mathrm{e}^{2\lambda t}\sigma\xi_t\, dB_t \end{aligned}$$

and again the terms in $dt$ vanish and the coefficient of $dB_t$ is a process in $M^2$, so that $Z$ is a (square integrable) martingale.

c) We can repeat the argument of b) for the function $u(x) = x^3 + cx$: we have

$$Lu(x) = -4\lambda x^3 - \lambda cx + 3\sigma^2 x$$

so that

$$\begin{gathered}
dS_t = \alpha e^{\alpha t} u(\xi_t)\,dt + e^{\alpha t}\,du(\xi_t) = \\
= \alpha e^{\alpha t} u(\xi_t)\,dt + e^{\alpha t} Lu(\xi_t)\,dt + u'(\xi_t)\sigma(\xi_t)\,dB_t = \\
\alpha e^{\alpha t}\big(\xi_t^3 + c\xi_t\big)\,dt + e^{\alpha t}\big(-3\lambda\xi_t^3 - \lambda c\xi_t + 3\sigma^2\xi_t\big) + \sigma(3\xi_t^2 + c)\,dB_t = \\
e^{\alpha t}\Big(\alpha\xi_t^3 + \alpha c\xi_t - 3\lambda\xi_t^3 - \lambda c\xi_t + 3\sigma^2\xi_t\Big)\,dt + e^{\alpha t}\sigma(3\xi_t^2 + c)\,dB_t = \\
e^{\alpha t}\Big((\alpha - 3\lambda)\xi_t^3 + (3\sigma^2 + \alpha c - \lambda c)\xi_t\Big)\,dt + \sigma(3\xi_t^2 + c)\,dB_t
\end{gathered}$$

In order for the term in $dt$ to vanish must be $\alpha = 3\lambda$ and $c = -\frac{3\sigma^2}{2\lambda}$.

**Exercise 3.** a) This is a particular case of a general situation seen in Exercise '`ex8.oubridge`' (coefficients depending on $t$ and linear on the state variable). The solution of the ''homogeneous equation''

$$d\xi_t = -\frac{1}{2}\,\frac{\xi_t}{1-t}\,dt, \qquad \xi_0 = x$$

is immediately found to be equal to

$$\xi_t = \sqrt{1-t}\,x\;,$$

and the ''particular solution'' is given by $t \mapsto \sqrt{1-t}\,C_t$, where $C$ must satisfy the equation

$$\sqrt{1-t}\,dC_t = \sqrt{1-t}\,dB_t$$

i.e. $C_t = B_t$ and the solution of (2) is

$$\xi_t = \sqrt{1-t}\,x + \sqrt{1-t}\,B_t\;.$$

$\xi$ is a Gaussian process, as we know that the stochastic integral with respect to a deterministic integrand always gives rise to a Gaussian process.

b) We have

$$\mathrm{Var}(\xi_t) = (1-t)t$$

which is the same as the variance of a Brownian bridge. As for the covariance function we have, for $s \le t$,

$$\mathrm{Cov}(\xi_t, \xi_s) = \mathrm{E}\big[\sqrt{1-t}\,B_t\,\sqrt{1-s}\,B_s\big] = s\sqrt{1-s}\sqrt{1-t}$$

which is different from the covariance function of a Brownian bridge. Remark that, if the starting point $x$ is the origin, then, for very $t, 0 \le t \le 1$, the distribution of $\xi_t$ coincides with the distribution of a Brownian bridge at time $t$, but $\xi$ is not a Brownian bridge.

**Exercise 4.** a) We have

$$Y_t^2 = \sum_{i \ne j} \alpha_i \alpha_j B_i(t) B_j(t) + \sum_i \alpha_i^2 B_i(t)^2 \ .$$

We know that, if $i \ne j$, $B_i B_j$ is a martingale and that $t \mapsto B_i(t)^2 - t$ is a martingale. Therefore

$$Y_t - \sum_i \alpha_i^2 \, t$$

is a martingale, which is the same a saying that $\langle Y \rangle_t = \sum_i \alpha_i^2 \, t$.

b) We can write

$$X_t = \mathrm{e}^{ct + Y_t}$$

hence by Ito's formula

$$dX_t = \mathrm{e}^{ct+Y_t} \big( c\, dt + dY_t \big) + \frac{1}{2}\, \mathrm{e}^{ct+Y_t} d\langle Y \rangle_t = X_t \Big( c + \sum_i \alpha_i^2 \Big)\, dt + X_t \sum_i \alpha_i \, dB_i(t)$$

If

$$W_t = \frac{1}{\sqrt{\sum_i \alpha_i^2}} \sum_i \alpha_i \, dB_i(t)$$

it is easy to check that $\mathrm{E}[W_t^2] = t$, as well as the other properties characterizing the Brownian motion (it is gaussian and with independent increments). Therefore we have (3) con $b(x) = x\Big( c + \sum_i \alpha_i^2 \Big)$ and $\sigma(x) = x \sqrt{\sum_i \alpha_i^2}$. In particular $X$ is a geometric Brownian motion.