UNIVERSITÀ DI ROMA TOR VERGATA

Laurea Magistrale in Matematica Pura e Applicata

Corso di *EP-Elementi di Probabilità*
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Exercise 1 Let $(B_1(t), B_2(t))$ a two-dimensional Brownian motion and let

$$Z_t = \int_0^t \frac{1}{1+4s}\, dB_1(s) \int_0^s \mathrm{e}^{-B_1(u)^2}\, dB_2(u)\ .$$

a) Is Z a martingale? Determine the processes $(\langle Z, B_1\rangle_t)_t$ and $(\langle Z, B_2\rangle_t)_t$.

b) Prove that the limit $Z_\infty = \lim_{t\to+\infty} Z_t$ exists a.s and in L^2 and compute $\mathrm{E}[Z_\infty]$ and $\mathrm{Var}(Z_\infty)$.

Exercise 2 Let, for $\alpha > 0$,

$$X_t = \sqrt{\alpha+1} \int_0^t u^{\alpha/2}\, dB_u\ .$$

a) Compute $\mathrm{P}(\sup_{s\le 2} X_s \ge 1)$.

b) Let $\tau = \inf\{t > 0;\, X_t \ge 1\}$. Compute the density of the r.v. τ. For which values of α does τ have finite expectation?

Exercise 3 Let $B = (B_1, B_2)$ a 2-dimensional Brownian motions.

a) Which one of the following is a Brownian motion?

$$\begin{aligned}
W_1(t) &= \int_0^t \sin B_2(s)\, dB_1(s) + \int_0^t \cos B_2(s)\, dB_1(s)\\
W_2(t) &= \int_0^t \sin B_2(s)\, dB_1(s) + \int_0^t \cos B_2(s)\, dB_2(s)\\
W_3(t) &= \int_0^t \cos B_2(s)\, dB_1(s) - \int_0^t \sin B_2(s)\, dB_2(s)\ .
\end{aligned}$$

b) Is $t \mapsto W_t := (W_2(t), W_3(t))$ a 2- dimensional Brownian motion?

Exercise 4 Let $B = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (B_t)_t, \mathrm{P})$ be a Brownian motion, $\theta \in \mathbb{R}$, $T > 0$.

a) Let $t \le T$. Which is the value of $\mathrm{E}[\mathrm{e}^{i\theta B_T} \,|\, \mathscr{F}_t]$? And of $\mathrm{E}[\sin(\theta B_T) \,|\, \mathscr{F}_t]$?

b) Determine a process $X \in M^2([0, T])$ such that

$$\sin(\theta B_T) = \int_0^T X_s\, dB_s$$

c) Compute

$$\int_0^1 \mathrm{e}^{-\frac{1}{2}(1-s)} \sin B_s \, dB_s$$

$$\int_0^1 \mathrm{e}^{-\frac{1}{2}(1-s)} \cos B_s \, dB_s \; .$$

Solutions

Exercise 1. a) Let

$$X_s = \int_0^s \mathrm{e}^{-B_1(u)^2}\, dB_2(s)\ .$$

We have, for every $t \ge 0$,

$$\mathrm{E}[X_t^2] = \mathrm{E}\Big[\int_0^t \mathrm{e}^{-2B_s^2}\, ds\Big] = \int_0^t \frac{1}{\sqrt{1+4s}}\, ds = \frac{1}{2}\big(\sqrt{1+4t}-1\big)\ .$$

As we can write

$$Z_t = \int_0^t \frac{X_s}{1+4s}\, dB_1(s) \tag{1}$$

and

$$\mathrm{E}\Big[\frac{X_s^2}{(1+4s)^2}\Big] = \frac{1}{2}\,\frac{\sqrt{1+4s}-1}{(1+4s)^2}$$

the integrand $s \mapsto \frac{X_s}{1+4s}$ is in M^2 and Z is a martingale.

From the notation (1) we have easily

$$\langle Z, B_2\rangle_t = 0, \quad \langle Z, B_1\rangle_t = \int_0^t \frac{X_s}{1+4s}\, ds\ .$$

b) It is sufficient (and also necessary...) to prove that Z is bounded in L^2. Now

$$\mathrm{E}[Z_t^2] = \int_0^t \frac{\mathrm{E}[X_s^2]}{(1+4s)^2}\, ds = \frac{1}{2}\int_0^t \Big(\frac{1}{(1+4s)^{3/2}} - \frac{1}{(1+4s)^2}\Big)\, ds\ .$$

The left most integral being convergent as $t \to +\infty$, the L^2 norm of Z is bounded, so that Z converges a.s. and in L^2. As L^2 convergence implies the convergence of the expectations we have immediately $\mathrm{E}[Z_\infty] = 0$. Also L^2 convergence implies the convergence of the second order moment, so that

$$\begin{aligned}\mathrm{E}[Z_\infty^2] &= \lim_{t\to+\infty} \mathrm{E}[Z_t^2] = \frac{1}{2}\int_0^{+\infty} \frac{1}{(1+4s)^{3/2}} - \frac{1}{(1+4s)^2}\, ds =\\ &= -2(1+4s)^{-1/2}\Big|_0^{+\infty} - 4(1+4s)^{-1}\Big|_0^{+\infty} = 6\ .\end{aligned}$$

Exercise 2. a) We know already that X is a time changed Brownian motion, more precisely

$$X_t = W_{A_t}$$

where

$$A_t = (\alpha+1)\int_0^t u^\alpha\,du = t^{\alpha+1}\ .$$

Therefore

$$\begin{aligned}\mathrm{P}\Big(\sup_{s\le 2} X_s\ge 1\Big) &= \mathrm{P}\Big(\sup_{s\le 2} W_{A_s}\ge 1\Big) = \mathrm{P}\Big(\sup_{t\le A(2)} W_t\ge 1\Big) = 2\mathrm{P}(W_{A(2)}\ge 1) =\\ &= 2\mathrm{P}(\sqrt{A(2)}\,W_1\ge 1) = 2\mathrm{P}(W_1\ge 2^{-\frac12\,(\alpha+1)}) =\\ &= \frac{2}{\sqrt{2\pi}}\int_{2^{-\frac12\,(\alpha+1)}}^\infty \mathrm{e}^{-t^2/2}\,dt\ .\end{aligned}$$

b) The previous computation with t instead of 2 gives

$$\mathrm{P}(\tau\le t) = \mathrm{P}\Big(\sup_{s\le t} X_s\ge 1\Big) = \frac{2}{\sqrt{2\pi}}\int_{t^{-\frac12\,(\alpha+1)}}^\infty \mathrm{e}^{-s^2/2}\,ds$$

and taking the derivative we find the density of τ:

$$f_\tau(t) = \frac{2}{\sqrt{2\pi}}\,\frac12\,(\alpha+1)t^{-\frac12\,(\alpha+1)-1}\mathrm{e}^{-1/(2t^{\alpha+1})} = \frac{(\alpha+1)}{\sqrt{2\pi\,t^{\alpha+3}}}\,\mathrm{e}^{-1/(2t^{\alpha+1})}\ .$$

In order for τ to have finite mathematical expectation $t\mapsto tf_\tau(t)$ must be integrable. Now at zero the integrand tends to 0 fast enough because of the exponential whereas at infinity $tf_\tau(t)\sim t^{1-\frac12\,(\alpha+3)}$. It is therefore necessary (and sufficient) that $1-\frac12\,(\alpha+3)<-1$, i.e. $\alpha>1$.

Exercise 3. a) We have

$$\langle W_1\rangle_t = \int_0^t(\cos B_2(s)+\sin B_2(s))^2\,ds = \int_0^t 1+\sin(2B_2(s))\,ds = t+\int_0^t\sin(2B_2(s))\,ds\ne t$$

so that W_1 is not a Brownian motion. Conversely, $H(s)=(\sin B_2(s),\cos B_2(s))$ is a 2-dimensional process having modulus equal to 1 and we can write

$$W_2(t) = \int_0^t H(s)\,dB_s$$

so that W_2 turns out to be a Brownian motion, thanks to Corollary 7.24 of class notes. The same argument gives that also W_3 is a Brownian motion.

b) We can write

$$d\begin{pmatrix}W_2(t)\\ W_3(t)\end{pmatrix} = \underbrace{\begin{pmatrix}\sin B_2(t) & \cos B_2(t)\\ \cos B_2(t) & -\sin B_2(t)\end{pmatrix}}_{=A}\begin{pmatrix}dB_1(t)\\ dB_2(t)\end{pmatrix}\ .$$

As the matrix A above is orthogonal, $W=(W_1,W_2)$ is a Brownian motion thanks to Proposition 7.26 of class notes.

Exercise 4. a) Recalling that $t\mapsto \mathrm{e}^{i\theta B_t+\frac12\theta^2 t}$ is a martingale, we have

$$\mathrm{E}[\mathrm{e}^{i\theta B_T}\,|\,\mathscr{F}_t]=\mathrm{E}[\mathrm{e}^{i\theta B_T+\frac12\theta^2 T}\,|\,\mathscr{F}_t]\,\mathrm{e}^{-\frac12\theta^2 T}=\mathrm{e}^{i\theta B_t+\frac12\theta^2 t}\mathrm{e}^{-\frac12\theta^2 T}=\mathrm{e}^{i\theta B_t-\frac12\theta^2(T-t)}\ .$$

We have

$$\mathrm{E}[\sin(\theta B_T)\,|\,\mathscr{F}_t]=\mathrm{E}[\mathrm{Im}(\mathrm{e}^{i\theta B_T})\,|\,\mathscr{F}_t]=\mathrm{Im}(\mathrm{e}^{i\theta B_t-\frac12\theta^2(T-t)})=\mathrm{e}^{-\frac12\theta^2(T-t)}\sin(\theta B_t)\ .$$

b) Let us compute the stochastic differential of $Z_t=\mathrm{e}^{-\frac12\theta^2(T-t)}\sin(\theta B_t)$. We can write $Z_t=f(B_t,t)$ with $f(x,t)=\mathrm{e}^{-\frac12\theta^2(T-t)}\sin(\theta x)$. We have

$$\begin{aligned}\frac{\partial f}{\partial t}(x,t)&=\frac12\,\theta^2\,\mathrm{e}^{-\frac12\theta^2(T-t)}\cos(\theta x)\\ \frac{\partial f}{\partial x}(x,t)&=\theta\mathrm{e}^{-\frac12\theta^2(T-t)}\cos(\theta x)\\ \frac{\partial^2 f}{\partial x^2}(x,t)&=-\theta^2\mathrm{e}^{-\frac12\theta^2(T-t)}\sin(\theta x)\ .\end{aligned}$$

and plugging these values into Ito's formula we find

$$dZ_t=\theta\mathrm{e}^{-\frac12\theta^2(T-t)}\cos(\theta B_t)\,dB_t$$

and therefore

$$\sin(\theta B_T)=Z_T-Z_0=\theta\int_0^T\mathrm{e}^{-\frac12\theta^2(T-t)}\cos(\theta B_t)\,dB_t$$

i.e. $X_t=\theta\mathrm{e}^{-\frac12\theta^2(T-t)}\cos(\theta B_t)$.

c) Form the computation of b) we have, with the choice $T=1,\theta=1$,

$$\int_0^1\mathrm{e}^{-\frac12(1-s)}\cos B_s\,dB_s=\sin B_1\ .$$

For the other integral we can repeat the argument above or apply Ito's formula in the following, equivalent, way, considering $t\mapsto\mathrm{e}^{-\frac12(1-t)}\cos B_t$ as the product of $t\mapsto\mathrm{e}^{-\frac12(1-t)}$ and of $t\mapsto\cos B_t$. We have then

$$d(\mathrm{e}^{-\frac12(1-t)}\cos B_t)=\mathrm{e}^{-\frac12(1-t)}\Big(-\sin B_t\,dB_t-\frac12\,\cos B_t\,dt\Big)+\frac12\,\mathrm{e}^{-\frac12(1-t)}\cos B_t\,dt$$

i.e.

$$d(\mathrm{e}^{-\frac12(1-t)}\cos B_t)=-\mathrm{e}^{-\frac12(1-t)}\sin B_t\,dB_t$$

which gives

$$\int_0^1\mathrm{e}^{-\frac12(1-t)}\sin B_t\,dB_t=-\mathrm{e}^{-\frac12(1-t)}\cos B_t\Big|_0^1=\mathrm{e}^{-\frac12}-\cos B_1\ .$$