UNIVERSITÀ DI ROMA TOR VERGATA

Laurea Magistrale in Matematica Pura e Applicata

Corso di *EP-Elementi di Probabilità*
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Exercise 1 Let
$$X_t = \int_0^t e^{-B_s^2}\, dB_s\ .$$
Prove that X is a square integrable martingale. Is it bounded in L^2?
(Recall that $E[e^{\alpha Z^2}] = \frac{1}{\sqrt{1-2\alpha}}$ for $Z \sim N(0,1)$).

Exercise 2 Let $B = (\Omega, \mathscr{F}, (\mathscr{G}_t)_t, (B_t)_t, P)$ be a natural Brownian motion and let $\widetilde{\mathscr{G}}_t = \mathscr{G}_t \vee \sigma(B_1)$.

a) Let s be fixed with $0 < s < t < 1$. Determine a square integrable function Φ and a number α (both possibly depending on s, t) such that the r.v.
$$B_t - \int_0^s \Phi(u)\, dB_u - \alpha B_1$$
is orthogonal to B_1 and to B_v for every $v \le s$.

b) Compute $E[B_t \,|\, \widetilde{\mathscr{G}}_s]$. Is B a Brownian motion also with respect to the filtration $(\widetilde{\mathscr{G}}_t)_t$?

Exercise 3 Let B be a Brownian motion. Compute the stochastic differential of $X_t = B_t^2 e^{B_t}$.

Exercise 4 a) Let $(B_t)_t$ a Brownian motion. Determine which of the following processes is a martingale
$$X_t = e^{t/2} \sin B_t$$
$$Y_t = e^{t/2} \cos B_t\ .$$
Compute $E(X_t)$ and $E(Y_t)$ for $t = 1$.

b) Prove that X and Y are Ito processes and compute $\langle X, Y\rangle_t$.

Exercise 5 Let B a Brownian motion.

a) For $n \ge 2$ write the stochastic differential of $X_t = B_t^n$.

b) Prove that
$$E[B_t^n] = \frac{1}{2}\, n(n-1) \int_0^t E[B_s^{n-2}]\, ds\ .$$

c) Recalling that for a $N(0,1)$-distributed r.v. we have $E[Z^4] = 3$, deduce the value of $E[Z^6]$.

Solutions

Exercise 1. We must check that $s \mapsto \mathrm{e}^{-B_s^2}$ is a process in M^2. Actually, for every $t \ge 0$,

$$\mathrm{E}\Big[\int_0^t \mathrm{e}^{-2B_s^2}\, ds\Big] = \int_0^t \mathrm{E}[\mathrm{e}^{-2B_s^2}]\, ds = \int_0^t \frac{1}{\sqrt{1+4s}}\, ds < +\infty$$

In order to investigate whether X is bounded in L^2, we just remark that

$$\mathrm{E}[X_t^2] = \mathrm{E}\Big[\int_0^t \mathrm{e}^{-2B_s^2}\, ds\Big] = \int_0^t \frac{1}{\sqrt{1+4s}}\, ds \quad \underset{t\to+\infty}{\to} \quad +\infty$$

so that the answer is no.

Exercise 2. a) Orthogonality with respect to B_v for $v \le s$ imposes the condition

$$0 = \mathrm{E}\Big[\Big(B_t - \int_0^s \Phi(u)\, dB_u - \alpha B_1\Big) B_v\Big] = v - \int_0^v \Phi(u)\, du - \alpha v$$

i.e.

$$v(1-\alpha) = \int_0^v \Phi(u)\, du, \qquad \text{for every } v \le s \tag{1}$$

and therefore $\Phi \equiv 1-\alpha$ on $[0, s]$. Orthogonality with respect to B_1 conversely requires

$$0 = \mathrm{E}\Big[\Big(B_t - \int_0^s \Phi(u)\, dB_u - \alpha B_1\Big) B_1\Big] = t - \int_0^s \Phi(u)\, du - \alpha$$

i.e., taking into account that $\Phi \equiv 1-\alpha$,

$$0 = t - (1-\alpha)s - \alpha = t - s - \alpha(1-s)$$

i.e.

$$\alpha = \frac{t-s}{1-s}, \qquad \Phi(u) \equiv \frac{1-t}{1-s} \cdot$$

b) If $X = \frac{t-s}{1-s}\, B_1 + \frac{1-t}{1-s}\, B_s$, in a) we have proved that the r.v. $B_t - X$, which is centered, is independent of $\widetilde{\mathscr{G}}_s$. As X is moreover $\widetilde{\mathscr{G}}_s$-measurable, we have

$$\mathrm{E}[B_t \,|\, \widetilde{\mathscr{G}}_s] = \mathrm{E}[(B_t - X) + X \,|\, \widetilde{\mathscr{G}}_s] = X + \mathrm{E}[B_t - X] = X = \frac{t-s}{1-s}\, B_1 + \frac{1-t}{1-s}\, B_s\ . \tag{2}$$

Remark that we have also $X = B_s + \frac{t-s}{1-s}\,(B_1 - B_s)$.

B is adapted to the filtration $(\mathscr{G}_t)_t$ but is not a $(\mathscr{G}_t)_t$-martingale, therefore cannot be a Brownian motion with respect to this filtration.

Exercise 3. First method. We have

$$
\begin{aligned}
d B_t^2 &= 2 B_t \, d B_t + dt \\
d \mathrm{e}^{B_t} &= \mathrm{e}^{B_t} \, d B_t + \frac{1}{2} \, \mathrm{e}^{B_t} \, dt
\end{aligned}
$$

and by the formula for the product

$$
\begin{aligned}
d X_t &= \mathrm{e}^{B_t} (2 B_t \, d B_t + dt) + B_t^2 \mathrm{e}^{B_t} (d B_t + \tfrac{1}{2} \, dt) + 2 B_t \mathrm{e}^{B_t} \, dt = \\
&= \mathrm{e}^{B_t} (2 B_t + B_t^2) \, d B_t + \mathrm{e}^{B_t} (\tfrac{1}{2} \, B_t^2 + 2 B_t + 1) \, dt
\end{aligned}
$$

Second method. We just apply Ito's formula to the Brownian motion and to the function $f(x) = x^2 \mathrm{e}^x$. We have

$$
\begin{aligned}
f'(x) &= \mathrm{e}^x (2x + x^2) \\
f''(x) &= \mathrm{e}^x (2x + x^2 + 2 + 2x) = \mathrm{e}^x (x^2 + 4x + 2) \, .
\end{aligned}
$$

Therefore

$$
d X_t = f'(B_t) \, d B_t + \frac{1}{2} \, f''(B_t) \, dt = \mathrm{e}^{B_t} (2 B_t + B_t^2) \, d B_t + \mathrm{e}^{B_t} (\tfrac{1}{2} \, B_t^2 + 2 B_t + 1) \, dt
$$

In this case this second method appears to be simpler.

Exercise 4. a) The simplest way is an application of Ito's formula. This can be done in many possible ways. For instance if $u(x, t) = \mathrm{e}^{t/2} \sin x$, we have

$$
d X_t = d u(B_t, t) = \frac{\partial u}{\partial t}(t, B_t) \, dt + \frac{\partial u}{\partial x}(t, B_t) \, d B_t + \frac{1}{2} \, \frac{\partial^2 u}{\partial x^2}(t, B_t) \, dt
$$

and as

$$
\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \, \mathrm{e}^{t/2} \sin x, \quad \frac{\partial u}{\partial x}(t, x) = \mathrm{e}^{t/2} \cos x, \quad \frac{\partial^2 u}{\partial x^2}(t, x) = -\mathrm{e}^{t/2} \sin x
$$

we find

$$
d X_t = \Big(\frac{1}{2} \, \mathrm{e}^{t/2} \sin B_t - \frac{1}{2} \, \mathrm{e}^{t/2} \sin B_t \Big) \, dt + \mathrm{e}^{t/2} \cos B_t \, d B_t = \mathrm{e}^{t/2} \cos B_t \, d B_t
$$

and therefore X is a local martingale and even a martingale, being bounded on bounded intervals. For Y the same computation with $u(x, t) = \mathrm{e}^{t/2} \cos x$ gives

$$
d Y_t = -\mathrm{e}^{t/2} \sin B_t \, d B_t \, .
$$

so also Y is a martingale. As the expectation of a martingale is constant we have $\mathrm{E}(X_t) = 0$, $\mathrm{E}(Y_t) = 1$ for every $t \geq 0$.

More quickly the imaginative reader might have observed that

$$Y_t + i X_t = \mathrm{e}^{i B_t + \frac{t}{2}}$$

Which is known to be an exponential complex martingale, hence both its real and imaginary parts are martingales.

b) The computation of the stochastic differentials of a) implies that both X and Y are Ito's processes. It is immediate that

$$\langle X, Y \rangle_t = -\int_0^t \mathrm{e}^s \cos B_s \sin B_s \, ds \ .$$

Exercise 5. a)

$$dB_t^n = n B_t^{n-1} \, dB_t + \frac{1}{2} n(n-1) B_t^{n-2} \, dt$$

b) Just take the expectation in the expression

$$B_t^n = n \int_0^t B_s^{n-1} \, dB_s + \frac{1}{2} n(n-1) \int_0^t B_s^{n-2} \, ds$$

and recognize that the stochastic integral is a martingale.

c) Choosing $n = 6$, we have

$$\mathrm{E}[B_t^6] = 15 \int_0^t \mathrm{E}[B_s^4] \, ds = 15 \cdot 3 \int_0^t s^2 \, ds = 15t^3$$

as actually $B_s^4 \sim s^2 B_1$. Taking $t = 1$ we find $\mathrm{E}[Z^6] = 15$.