UNIVERSITÀ DI ROMA TOR VERGATA

Laurea Magistrale in Matematica Pura e Applicata

Corso di *EP-Elementi di Probabilità*
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Exercise 1 Let B a Brownian motion and let, for $t \ge 0$,

$$Z_t = \frac{1}{\sqrt{1+t}}\, B_{1+t}\ .$$

a) Is it a Markov process? Which is its transition function? Which is its initial distribution?

b) Is it time homogeneous? Prove that it is not a stationary process.

c) Let $(\Omega, \mathscr{F}, (\mathscr{F}_s)_s, (Z_t)_t, (\mathrm{P}^{x,s})_{x,s})$ a realization of the Markov process associated to the transition function computed in a). Prove that, whatever the starting point x and the initial time s,

$$Z_t \underset{t\to+\infty}{\to} N(0,1) \qquad \text{in law}\ .$$

Exercise 2 Let $B = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (B_t)_t, \mathrm{P})$ be a m-dimensional Brownian motion and let $X_t = B_{ct}$, where $c > 0$.

a) Prove that X is a time-homogeneous Markov process and determine its transition function p.

b1) Let $X = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (X_t)_t, (\mathrm{P}^{x,s})_x)$ the Markov process associated to function p. Prove that the distribution of X_t under $\mathrm{P}^{x,s}$ is the same as the distribution of $x + B_{c(t-s)}$.

b2) Which is the generator of X?

Exercise 3 Let $X = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (X_t)_t, \mathrm{P})$ be a time homogeneous Markov process taking values in some measurable space $(E, \mathscr{E})$ and associated to a transition function p. Let $A \in \mathscr{E}$.

a) For $s \le t$ compute

$$\mathrm{E}[1_A(X_t)\,|\,\mathscr{F}_s]$$

and express it as a function of s, X_s and p.

b) Let, for $s \le t$,

$$M_s = p(t-s, X_s, A)\ .$$

b1) Prove that $(M_s)_{s\le t}$ is a $(\mathscr{F}_t)_{s\le t}$-martingale. Is it uniformly integrable?

c) Assume in addition that X is a Brownian motion.

c1) Write explicitly the expression of $p(t-s, x, A)$. Remark that for $s < t$ this is a continuous function of (s, x) (just give an argument, I do not ask for a complete proof). Deduce that M is a continuous martingale. Prove that it converges a.s. as $s \nearrow t$.

c2) Let $\mathscr{F}_{t-} = \bigvee_{s<t} \mathscr{F}_s$ the σ-algebra generated by $\bigcup_{s<t} \mathscr{F}_s$. Let $M_\infty = \lim_{s\uparrow t} M_s$. Prove that $M_\infty = \mathrm{E}[1_A(X_t)\,|\,\mathscr{F}_{t-}]$ (Hint: recall the proof of Proposition 4.21).

c3) Assume that $(\mathscr{F}_t)_t$ is augmented (i.e. that for every t $\mathscr{F}_t$ contains the negligible events of $\mathscr{F}$). Prove that $\lim_{s\uparrow t} M_s = 1_A(X_t)$ a.s.

Solutions

Exercise 1. a) The covariance function of Z is, for $s \le t$,

$$K_{t,s} = \mathrm{Cov}(X_s, X_t) = \frac{1}{\sqrt{1+t}\sqrt{1+s}} \mathrm{E}[B_{1+t} B_{1+s}] = \frac{1+s}{\sqrt{1+t}\sqrt{1+s}} = \frac{\sqrt{1+s}}{\sqrt{1+t}}$$

and we have, for $u \le s \le t$

$$\begin{aligned} K_{u,t} &= \frac{\sqrt{1+u}}{\sqrt{1+t}}, \\ K_{u,s} K_{s,s}^{-1} K_{s,t} &= \frac{\sqrt{1+s}}{\sqrt{1+t}} \frac{\sqrt{1+u}}{\sqrt{1+s}} = \frac{\sqrt{1+u}}{\sqrt{1+t}} \end{aligned}$$

so that this is a Markov process. Its transition function $p(s,t,x,dy)$, $s \le t$, is the conditional distribution of X_t given $X_s = x$, which is of course Gaussian with mean

$$\mathrm{E}(X_t) + \frac{K_{t,s}}{K_{s,s}} (x - \mathrm{E}(X_s)) = \frac{\sqrt{1+s}}{\sqrt{1+t}}\, x$$

and variance

$$\mathrm{Var}(X_t) - \frac{K_{t,s}^2}{K_{s,s}} = 1 - \frac{1+s}{1+t} = \frac{t-s}{1+t}$$

i.e. $p(s,t,x,dy) \sim N(\frac{\sqrt{1+s}}{\sqrt{1+t}}\, x, \frac{t-s}{1+t})$. The initial distribution is the distribution of $B_1 \sim N(0,1)$.

b) As the law $p(s,t,x,dy)$ does depend on the couple s,t and not just on the difference $t-s$, Z is not time homogeneous and *a fortiori* is not stationary (the joint distributions of (X_s, X_t) and (X_{s+h}, X_{t+h}) are different as they have different covariance matrices).

c) The law of Z_t with starting point x and initial time s is $p(s,t,x,dy) \sim N(\frac{\sqrt{1+s}}{\sqrt{1+t}}\, x, \frac{t-s}{1+t})$. As for $t \to +\infty$ the mean converges to 0 and the variance to 1, Z_t converges in law to a $N(0,1)$ distribution for every initial conditions x, s.

Exercise `monique1`. a) We have

$$\mathrm{E}[X_t \,|\, \mathscr{F}_s] = \mathrm{E}[ab + (b-a)B_t - B_t^2 + t \,|\, \mathscr{F}_s] = ab - (b-a)B_s - \mathrm{E}[((B_t - B_s) + B_s)^2 \,|\, \mathscr{F}_s]$$

Now

$$\begin{aligned} \mathrm{E}[((B_t - B_s) + B_s)^2 \,|\, \mathscr{F}_s] &= \mathrm{E}[(B_t - B_s)^2 + B_s^2 - 2B_s(B_t - B_s) \,|\, \mathscr{F}_s] = \\ &= t - s + B_s^2 - 2B_s \underbrace{\mathrm{E}[B_t - B_s \,|\, \mathscr{F}_s]}_{=0} = t - s + B_s^2 \end{aligned}$$

and substituting we find $\mathrm{E}[X_t \,|\, \mathscr{F}_s] = X_s$.

b) First of all remark that τ is finite, thanks to the Iterated Logarithm Law. Then by the stopping theorem we have, for every $n > 0$,

$$ab = \mathrm{E}(X_0) = \mathrm{E}(X_{\tau\wedge n}) = (b - B_{\tau\wedge n})(a + B_{\tau\wedge n}) + \mathrm{E}(\tau \wedge n)$$

Now the quantity $(b - B_{\tau\wedge n})(a + B_{\tau\wedge n})$ is bounded in n as $-a \le B_{\tau\wedge n} \le b$. As $B_{\tau\wedge n} \to B_\tau$ as $n \to \infty$ and B_τ can only take the values $-a$ or b, we have $(b - B_{\tau\wedge n})(a + B_{\tau\wedge n}) \to 0$ as $n \to \infty$ and also $\mathrm{E}[(b - B_{\tau\wedge n})(a + B_{\tau\wedge n})] \to 0$ by Lebesgue's theorem. Finally $\mathrm{E}(\tau \wedge n) \to \mathrm{E}(\tau)$ by Beppo Levy's theorem so that we have

$$\mathrm{E}(\tau) = ab\ .$$

Exercise 2. a) X being clearly a Gaussian process we just need to check the relation $K_{u,t} = K_{u,s}K_{s,s}^{-1}K_{s,t}$ for $u \le s \le t$, which is immediate as, for $s \le t$,

$$K_{s,t} = \mathrm{Cov}(B_{cs}, B_{ct}) = cs\ .$$

The transition probability $p(t - s, x, dy)$ is given by the conditional distribution of X_t given $X_s = x$, which is Gaussian with mean

$$\mathrm{E}(X_t) + \frac{K_{s,t}}{K_{s,s}}(x - \mathrm{E}(X_s)) = x$$

and variance

$$K_{t,t} - \frac{K_{s,t}^2}{K_{s,s}} = c(t - s)$$

i.e. $p(t, x, dy) \sim N(x, ct)$.

b1) This is immediate, as the distribution of X_t starting at x is $p(t, x, dy)$, which was just computed and that is a $N(x, ct)$ distribution, that coincides with the distribution of $x + B_{ct}$.

b2) We must compute

$$Lf(x) = \lim_{h\to 0} \frac{1}{h}\big(\mathrm{E}^x[f(X_h)] - f(x)\big) = \lim_{h\to 0} \frac{1}{h}\big(\mathrm{E}[f(x + B_{ch})] - f(x)\big)$$

We know that, if f is bounded and twice differentiable,

$$\lim_{h\to 0} \frac{1}{h}\big(\mathrm{E}[f(x + B_h)] - f(x)\big) = \frac{1}{2}\triangle f(x)$$

(the generator of the Brownian motion). Therefore, for every f bounded and twice differentiable,

$$Lf(x) = \frac{c}{2}\triangle f(x)$$

Exercise 3. a) By the Markov property

$$\mathrm{E}[1_A(X_t)\,|\,\mathscr{F}_s] = p(t-s, X_s, A)\ . \tag{1}$$

b1) From a) we know that $M_s = \mathrm{E}[1_A(X_t)\,|\,\mathscr{F}_s]$, so that the martingale property is immediate. Also uniform integrability is immediate from (1) and Proposition 4.17.

c1) We have

$$p(t-s, x, A) = \frac{1}{\sqrt{2\pi(t-s)}}\int_A \exp\Big(-\frac{(y-x)^2}{2(t-s)}\Big)\,dy$$

and the continuity in (s, x) follows from the usual results about integrals depending on a parameter. As a consequence $s \mapsto M_s$ turns out to be a composition of continuous functions, thus continuous itself. The existence of the a.s. and L^1 limit is a consequence of Theorem 4.24, as seen in class.

c2) We must prove that we have

$$\mathrm{E}[M_\infty 1_G] = \mathrm{E}[1_A(X_t)1_G] \tag{2}$$

for every set $G \in \mathscr{F}_{t-}$, or at least for G in a family $\mathscr{C} \subset \mathscr{F}_{t-}$ that is stable with respect to finite intersections and generating $\mathscr{F}_{t-}$. A convenient family is $\mathscr{C} = \bigcup_{s<t}\mathscr{F}_s$. Now if $G \in \mathscr{C}$, then $G \in \mathscr{F}_s$ for some s and

$$\mathrm{E}[M_\infty 1_G] = \lim_{u\to t}\mathrm{E}\big[\mathrm{E}[1_A(X_t)\,|\,\mathscr{F}_u]1_G\big] = \lim_{u\to t}\mathrm{E}\big[\mathrm{E}[1_A(X_t)1_G\,|\,\mathscr{F}_u]\big] = \mathrm{E}[1_A(X_t)1_G]$$

where we have used the fact that $\mathrm{E}[1_A(X_t)\,|\,\mathscr{F}_u]1_G = \mathrm{E}[1_A(X_t)1_G\,|\,\mathscr{F}_u]$, a soon as $u \ge s$. Therefore (2) is proved.

c3) This would be immediate from c2) if we knew that the r.v. $1_A(X_t)$ is $\mathscr{F}_{t-}$-measurable (by now we only know that it is $\mathscr{F}_t$-measurable).

But the filtration $(\mathscr{F}_s)_s$ certainly contains the natural filtration $(\mathscr{G}_s)_s$ of the Brownian motion. Hence also the augmented natural filtration $(\overline{\mathscr{G}}_s)_s$. Therefore, finally, just remark that $\mathscr{F}_{t-}$ contains $\overline{\mathscr{G}}_{t-}$ which coincides with $\overline{\mathscr{G}}_t$ by the fundamental property of continuity of the augmented natural filtration (Proposition 3.3). As $1_A(X_t)$ is $\overline{\mathscr{G}}_t$-measurable, then it is also $\mathscr{F}_{t-}$-measurable which allows to conclude.