UNIVERSITÀ DI ROMA TOR VERGATA

Laurea Magistrale in Matematica Pura e Applicata

Corso di *EP-Elementi di Probabilità*
P.Baldi
Tutorato 3 dell'11 aprile 2016

Exercise 1 Let B be a Brownian motion (with respect to a filtration $(\mathscr{F}_t)_t$).

a) Prove that, for every $\lambda \in \mathbb{R}$,

$$Y_t = e^{i\lambda B_t + \frac{1}{2}\lambda^2 t}$$

is a martingale.

b) Prove that, for every $\lambda \in \mathbb{R}$,

$$X_t = \cos(\lambda B_t)\, e^{\frac{1}{2}\lambda^2 t}$$

is a $(\mathscr{F}_t)_t$-martingale. Is it uniformly integrable?

Exercise 2 a) Let $B = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (B_t)_t, \mathrm{P})$ be a m-dimensional Brownian motion. Prove that, if $i \neq j$, the process $(B_i(t)B_j(t))_t$ is a $(\mathscr{F}_t)_t$-martingale.

b) Let $(M_t)_t$, $(N_t)_t$ be martingales on the same probability space $(\Omega, \mathscr{F}, \mathrm{P})$, with respect to the filtrations $(\mathscr{M}_t)_t$, $(\mathscr{N}_t)_t$ respectively. Let us assume moreover that the filtrations $(\mathscr{M}_t)_t$ and $(\mathscr{N}_t)_t$ are independent. Then the product $(M_t N_t)_t$ is a martingale of the filtration $\mathscr{H}_t = \mathscr{M}_t \vee \mathscr{N}_t$.

Exercise 3 Let B be a m-dimensional Brownian motion.

a) Prove that

$$X_t = |B_t|^2 - mt$$

is a martingale.

b) Let us denote by τ the exit time of B from the unit ball of $\mathbb{R}^m$. Compute $\mathrm{E}[\tau]$.

Exercise 4 Let B a Brownian motion. Recall that the process

$$X_t = B_t - tB_1, \quad 0 \le t \le 1 \tag{1}$$

is called a Brownian bridge and that, for $0 \le s \le t \le 1$, $\mathrm{E}(X_s X_t) = s(1-t)$.

a) Let, for $0 \le t \le 1$,

$$Z_t = \begin{cases} (1-t)B_{\frac{t}{1-t}} & 0 \le t < 1 \\ 0 & \text{if } t = 1 \end{cases}$$

Prove that Z is also a Brownian bridge, i.e. that it is equivalent to X. Is it continuous at $t = 1$?

c) Prove that there exists a Brownian motion W such that the process X of (1) is, for $0 \le t < 1$, of the form

$$X_t = (1-t)W_{\frac{t}{1-t}} \tag{2}$$

b) Prove that, for every $a > 0$,

$$\mathrm{P}\Big(\sup_{0\le t\le 1} X_t > a\Big) = \mathrm{P}\Big(\sup_{s>0} B_s - as > a\Big)$$

and deduce partition function and density of the r.v. $\sup_{0\le t\le 1} X_t$. [Remember Exercise 4.13]

Solutions

Exercise 1. a) In way similar to what already seen in class, we can show that Y is a (complex) $(\mathscr{F}_t)_t$-martingale:

$$\begin{aligned}\mathrm{E}(Y_t \mid \mathscr{F}_s) = \mathrm{e}^{\frac{1}{2}\lambda^2 t}\mathrm{E}(\mathrm{e}^{i\lambda B_s}\mathrm{e}^{i\lambda(B_t-B_s)} \mid \mathscr{F}_s) = \mathrm{e}^{\frac{1}{2}\lambda^2 t}\mathrm{e}^{i\lambda B_s}\mathrm{E}(\mathrm{e}^{i\lambda(B_t-B_s)}) =\\ = \mathrm{e}^{\frac{1}{2}\lambda^2 t}\mathrm{e}^{i\lambda B_s}\mathrm{e}^{-\frac{1}{2}\lambda^2(t-s)} = Y_s\ .\end{aligned}$$

b) The result of a) implies that the real part of Y is itself a martingale and remark now that $\mathrm{Re}\, Y_t = \cos(\lambda B_t)\,\mathrm{e}^{\frac{1}{2}\lambda^2 t}$.

In order to investigate uniform integrability, remark that there exists a sequence of times $(t_n)_n$ such that $t_n \to +\infty$ and $B_{t_n} = 0$. Hence $Y_{t_n} = \mathrm{e}^{\frac{1}{2}\lambda^2 t_n} \to +\infty$. If Y was uniformly integrable, then it would converge a.s. and in L^1, but this is impossible as we have just seen that the limit, if it existed, would be equal to $+\infty$, whereas $\mathrm{E}[Y_t] = \mathrm{E}[Y_0] = 1$.

Exercise 3. a) The simplest way is to remark that we already know that, if W is a real Brownian motion then $t \mapsto W_t^2 - t$ is a martingale. Then just remark that

$$X_t = B_1(t)^2 - t + B_2(t)^2 - t + \ldots + B_m(t)^2 - t$$

so that X appears to be a sum of martingales.

b) If we could apply the stopping theorem to the process X and to the stopping time τ, which unfortunately is not known to be bounded (and it is not) we would have

$$0 = \mathrm{E}[X_\tau] = \mathrm{E}[|B_\tau|^2] - m\mathrm{E}[\tau]\ . \tag{3}$$

Now obviously $|B_\tau| = 1$ a.s. so that from (3) we deduce

$$\mathrm{E}[\tau] = \frac{1}{m}$$

Let us prove (3). We have, for every $t \geq 0$, $0 = \mathrm{E}[X_{t\wedge\tau}] = \mathrm{E}[|B_{t\wedge\tau}|^2] - m\mathrm{E}[t \wedge \tau]$ by the stopping theorem applied to the stopping time $t \wedge \tau$. Hence

$$\mathrm{E}[|B_{t\wedge\tau}|^2] = m\mathrm{E}[t \wedge \tau]$$

Now we can just take the limit as $t \to +\infty$. The left-hand term converges to $\mathrm{E}[|B_\tau|^2]$ by Lebesgue's theorem (we have obviously $|B_{t\wedge\tau}|^2 \leq 1$, whereas the right-hand one increases to $\mathrm{E}[\tau]$ by Beppo Levi's theorem.

Exercise 2. a)Using the usual method of factoring out the increment, we have for $s \leq t$

$$\begin{aligned}\mathrm{E}\big[B_i(t)B_j(t) \mid \mathscr{F}_s\big] = \mathrm{E}\big[(B_i(s) + (B_i(t) - B_i(s))(B_j(s) + (B_j(t) - B_j(s)) \mid \mathscr{F}_s\big] =\\ = \mathrm{E}\big[B_i(s)B_j(s) + B_i(s)(B_j(t) - B_j(s)) + B_j(s)(B_i(t) - B_i(s)) + (B_i(t) - B_i(s))(B_j(t) - B_j(s)) \mid \mathscr{F}_s\big]\end{aligned}$$

Now just remark that, the increments being independent of $\mathscr{F}_s$,

$$\begin{aligned}
&\mathrm{E}\big[B_i(s)(B_j(t)-B_j(s))\,|\,\mathscr{F}_s\big]=B_i(s)\mathrm{E}\big[B_j(t)-B_j(s)\,|\,\mathscr{F}_s\big]=0\\
&\mathrm{E}\big[B_j(s)(B_i(t)-B_i(s))\,|\,\mathscr{F}_s\big]=B_j(s)\mathrm{E}\big[B_i(t)-B_i(s)\,|\,\mathscr{F}_s\big]=0\\
&\mathrm{E}\big[(B_i(t)-B_i(s))(B_j(t)-B_j(s))\,|\,\mathscr{F}_s\big]=\mathrm{E}\big[(B_i(t)-B_i(s))(B_j(t)-B_j(s))\big]=0
\end{aligned}$$

and therefore $\mathrm{E}[B_i(t)B_j(t)\,|\,\mathscr{F}_s]=B_i(s)B_j(s)$.

b) We must prove that, if $s\le t$,

$$\mathrm{E}[M_tN_t1_A]=\mathrm{E}[M_sN_s1_A]$$

for every $A\in\mathscr{H}_s$ or at least for every A in a subclass $\mathscr{C}_s\subset\mathscr{H}_s$, which generates $\mathscr{H}_s$ and is stable with respect to finite intersections (this is Remark ‘`cap3-rem31`’). One can consider the class of the events of the form $A_1\cap A_2$ with $A_1\in\mathscr{M}_s$, $A_2\in\mathscr{N}_s$. Actually this class is stable with respect to finite intersections and contains both $\mathscr{M}_s$ (choosing $A_2=\Omega$) and $\mathscr{N}_s$ (with $A_1=\Omega$). We have then, as the r.v.'s $M_t1_{A_1}$ and $N_t1_{A_2}$ are independent (the first one is $\mathscr{M}_t$-measurable whereas the second one is $\mathscr{N}_t$-measurable)

$$\begin{aligned}
\mathrm{E}[M_tN_t1_{A_1\cap A_2}]&=\mathrm{E}[M_t1_{A_1}N_t1_{A_2}]=\mathrm{E}[M_t1_{A_1}]\mathrm{E}[N_t1_{A_2}]=\mathrm{E}[M_s1_{A_1}]\mathrm{E}[N_s1_{A_2}]=\\
&=\mathrm{E}[M_s1_{A_1}N_s1_{A_2}]=\mathrm{E}[M_sN_s1_{A_1\cap A_2}]\,.
\end{aligned}$$

Exercise 4. a) Recall that the Brownian bridge X is a process that is centered, Gaussian and with covariance $\mathrm{Cov}(X_s,X_t)=s(1-t)$ for $0\le s\le t\le 1$. In order to prove that Z is equivalent to X it is therefore sufficient to verify that it also enjoys these properties. Actually it is immediate that it is Gaussian (the joint distributions are linear combinations of the joint distributions of B) and centered. Let $s\le t$, then

$$\mathrm{Cov}(Z_s,Z_t)=\mathrm{E}[Z_sZ_t]=(1-s)(1-t)\mathrm{E}\big[B_{\frac t{1-t}}B_{\frac s{1-s}}\big]\,.$$

The function $t\to\frac t{1-t}$ being increasing, the smallest between $\frac t{1-t}$ and $\frac s{1-s}$ is the last one, so that

$$\mathrm{Cov}(Z_s,Z_t)=(1-s)(1-t)\,\frac{s}{1-s}=s(1-t)\,.$$

The process Z having the same covariance function as X has also the same finite dimensional distributions and is therefore equivalent to it.

We still have to prove the continuity at $t=1$. Is it true that

$$\lim_{t\to1-}Z_t=0\,?$$

As $t\to1$ $\frac t{1-t}\to+\infty$ and by the iterated logarithm law we know that

$$\Big|(1-t)B_{\frac t{1-t}}\Big|\le(1-t)\sqrt{(2+\varepsilon)\frac{t}{1-t}\log\log\frac{t}{1-t}}$$

and it is now easy to see that the left-hand side goes to 0 as $t \to 1-$.

b) If a process W satisfying (2) existed then, substituting $u = \frac{t}{1-t}$, i.e. $t = \frac{u}{u+1}$, we would have

$$W_u = (1+u)X_{\frac{u}{u+1}} \tag{4}$$

We therefore have only to check that if X is a Brownian bridge, then W defined in (4) is a Brownian motion. It is clearly centered, Gaussian and continuous. We have also, for $s \le t$,

$$\mathrm{E}[W_s W_t] = (1+s)(1+t)\mathrm{E}\Big[X_{\frac{s}{s+1}} X_{\frac{t}{t+1}}\Big]\,.$$

Now the smallest between $\frac{s}{s+1}$ and $\frac{t}{t+1}$ is the first one, so that, recalling the covariance function of the Brownian bridge,

$$\mathrm{E}[W_s W_t] = (1+s)(1+t)\frac{s}{s+1}\Big(1-\frac{t}{t+1}\Big) = s$$

i.e. W is a Brownian motion.

b) We have

$$\mathrm{P}\Big(\sup_{0\le t\le 1} X_t > a\Big) = \mathrm{P}\Big(\sup_{0\le t<1} (1-t)B_{\frac{t}{1-t}} > a\Big) = \mathrm{P}\Big(\sup_{s>0} \frac{1}{s+1} B_s > a\Big) =$$
$$= \mathrm{P}\Big(\sup_{s>0} \frac{1}{s+1}(B_s - (s+1)a) > 0\Big) = \mathrm{P}\Big(\sup_{s>0} B_s - sa > a\Big)\,.$$

In Exercise 4.13 we have seen that the r.v. $\sup_{s>0} B_s - sa$ has an exponential law of parameter $2a$. Therefore

$$\mathrm{P}\Big(\sup_{0\le t\le 1} X_t > a\Big) = \mathrm{e}^{-2a^2}$$

and the partition function of the r.v. $\sup_{0\le t\le 1} X_t$ is $F(x) = 1 - \mathrm{e}^{-2x^2}$ for $x > 0$. Taking the derivative, the corresponding density is $f(x) = 4x\mathrm{e}^{-2x^2}$, still for $x \ge 0$.