UNIVERSITÀ DI ROMA TOR VERGATA

Laurea Magistrale in Matematica Pura e Applicata

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Exercise 1 Let B be a Brownian motion and let $\varepsilon > 0$.

a) Prove that

$$\mathrm{E}\Big[\Big(\int_t^{t+\varepsilon} B_u\, du\Big)^2\Big] = t\varepsilon^2 + \frac{1}{3}\,\varepsilon^3\ . \tag{1}$$

b) Which is the joint distribution of B_t and

$$Y_\varepsilon = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} B_u\, du\,?$$

c) Which is the conditional expectation of B_t given $Y_\varepsilon = y$? Compute its limit as $\varepsilon \to 0$.

d) Which is the variance of the conditional law of B_t given $Y_\varepsilon = y$? Compute its limit as $\varepsilon \to 0$.

Exercise 2 Let $(M_n)_n$ be a martingale with respect to a filtration $(\mathscr{F}_n)_n$.

a) Prove that, if $m \le n$, $\mathrm{E}[(M_n - M_m)^2] = \mathrm{E}[M_n^2] - \mathrm{E}[M_m^2]$.

b) Prove that if $(M_n^2)_n$ is also a martingale with respect to $(\mathscr{F}_n)_n$, then $(M_n)_n$ is constant.

Exercise 3 Let $(Y_n)_n$ be a sequence of i.i.d. r.v.'s such that $\mathrm{P}(Y_i = 1) = p$, $\mathrm{P}(Y_i = -1) = q$ with $q > p$. Let $X_n = Y_1 + \ldots + Y_n$.

a) Compute $\lim_{n\to\infty} \frac{1}{n}\, X_n$ and show that $\lim_{n\to\infty} X_n = -\infty$ a.s.

b) Show that

$$Z_n = \big(\tfrac{q}{p}\big)^{X_n}$$

is a martingale.

c) Let $a, b \in \mathbb{N}$ be positive numbers and let $\tau = \inf\{n,\, X_n = b \text{ or } X_n = -a\}$. Which is the value of $\mathrm{E}[Z_{n\wedge\tau}]$? Which is the value of $\mathrm{E}[Z_\tau]$?

d) Which is the value of $\mathrm{P}(X_\tau = b)$? (I.e. which is the probability for the random walk $(X_n)_n$ to exit from the interval $]-a, b[$ at b?)

Exercise 4 Let $B = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (B_t)_t, \mathrm{P})$ be a Brownian motion and $\lambda \in \mathbb{R}$.

a) Prove that

$$M_t = \mathrm{e}^{\lambda t} B_t - \lambda \int_0^t \mathrm{e}^{\lambda u} B_u\, du$$

is a martingale.

b) Prove that M has independent increments with respect to $(\mathscr{F}_t)_t$.

Solutions

Exercise 1. a) We have

$$\mathrm{E}\Big[\Big(\int_t^{t+\varepsilon} B_u\,du\Big)^2\Big]=\mathrm{E}\Big[\int_t^{t+\varepsilon} B_u\,du\int_t^{t+\varepsilon} B_v\,dv\Big]=\int_t^{t+\varepsilon} dv\int_t^{t+\varepsilon}\mathrm{E}][B_uB_v]\,du=$$
$$=\int_t^{t+\varepsilon} dv\int_t^{t+\varepsilon} u\wedge v\,du=\int_t^{t+\varepsilon} dv\int_t^{v} u\,du+\int_t^{t+\varepsilon} dv\int_v^{t+\varepsilon} u\,du$$

Now

$$\int_t^{t+\varepsilon} dv\int_t^{v} u\,du=\int_t^{t+\varepsilon}(\tfrac{v^2}{2}-\tfrac{t^2}{2})\,dv=\frac16\big((t+\varepsilon)^3-t^3\big)-\frac12\,t^2\varepsilon=$$
$$=\frac16\,(3t^2\varepsilon+3t\varepsilon^2+\varepsilon^3)-\frac12\,t^2\varepsilon=\frac12\,t\varepsilon^2+\frac16\,\varepsilon^3$$

The other integral at the right-hand side above can be computed similarly and gives the same contribution so that (1) is verified.

b) We know that this joint law is Gaussian. As both these r.v.'s are centered, in order to completely determine the distribution we only have to compute the covariance matrix. We know that the variance of B_t is equal to t. Thanks to a) the variance of Y_ε is equal to $t+\frac13\,\varepsilon$. For the covariance of B_t and Y_ε we have

$$\mathrm{E}\Big[B_t\,\frac1\varepsilon\int_t^{t+\varepsilon} B_u\,du\Big]=\frac1\varepsilon\int_t^{t+\varepsilon}\mathrm{E}[B_tB_u]\,du=\frac1\varepsilon\int_t^{t+\varepsilon} t\,du=t\ .$$

Hence the covariance matrix of (B_t,Y_ε) is

$$C=\begin{pmatrix} t & t\\ t & t+\frac13\,\varepsilon\end{pmatrix}$$

c) The conditional expectation is equal to

$$\mathrm{E}[B_t]+\frac{\mathrm{Cov}(B_t,Y_\varepsilon)}{\mathrm{Var}(Y_\varepsilon)}\,(y-\mathrm{E}[Y_\varepsilon])=\frac{t}{t+\frac13\varepsilon}\,y$$

and converges to y as $\varepsilon\to0$.

d) The variance of the conditional law of B_t given $Y_\varepsilon=y$ is

$$\mathrm{Var}(B_t)-\frac{\mathrm{Cov}(B_t,Y_\varepsilon)^2}{\mathrm{Var}(Y_\varepsilon}=t-\frac{t^2}{t+\frac13\,\varepsilon}=\frac{\frac13\,\varepsilon t}{t+\frac13\,\varepsilon}$$

and clearly converges to 0 as $\varepsilon\to0$.

Exercise 2. a) We have

$$\mathrm{E}[(M_n - M_m)^2] = \mathrm{E}[M_n^2 + M_m^2 - 2M_nM_m] = \mathrm{E}[M_n^2] + \mathrm{E}[M_m^2] - 2\mathrm{E}[M_nM_m]$$

but $\mathrm{E}[M_nM_m] = \mathrm{E}[\mathrm{E}[M_nM_m \,|\, \mathscr{F}_m]] = \mathrm{E}[M_m\mathrm{E}[M_n \,|\, \mathscr{F}_m]] = \mathrm{E}[M_m^2]$.

b) If $(M_n^2)_n$ is also a martingale with respect to $(\mathscr{F}_n)_n$, then, if $m \le n$,

$$\mathrm{E}[M_n^2] - \mathrm{E}[M_m^2] = 0 \ .$$

But then, thanks to a), also $\mathrm{E}[(M_n - M_m)^2] = 0$ which implies that $M_n = M_m$ a.s.

Exercise 3. a) Thanks to the law of large numbers we have a.s.

$$\tfrac{1}{n}X_n = \tfrac{1}{n}\,(Y_1 + \ldots + Y_n) \underset{n\to\infty}{\to} \mathrm{E}[Y_1] = p - q < 0$$

and it follows easily that $X_n \to -\infty$ a.s. Actually just remark that, a.s., $\frac{1}{n}X_n \le \frac{1}{2}\,(p-q) < 0$ for $n \ge n_0$.

b) We have, Y_{n+1} being independent of $\mathscr{F}_n$,

$$\begin{aligned}\mathrm{E}[Z_{n+1} \,|\, \mathscr{F}_n] &= \mathrm{E}[(\tfrac{q}{p})^{X_{n+1}} \,|\, \mathscr{F}_n] = \mathrm{E}[(\tfrac{q}{p})^{X_n+Y_{n+1}} \,|\, \mathscr{F}_n] = (\tfrac{q}{p})^{X_n}\mathrm{E}[(\tfrac{q}{p})^{Y_{n+1}} \,|\, \mathscr{F}_n] =\\ &= (\tfrac{q}{p})^{X_n}\mathrm{E}[(\tfrac{q}{p})^{Y_{n+1}}] = (\tfrac{q}{p})^{X_n}((\tfrac{q}{p})\mathrm{P}(Y=1) + (\tfrac{q}{p})^{-1}\mathrm{P}(Y=-1)) =\\ &= (\tfrac{q}{p})^{X_n}((\tfrac{q}{p})p + (\tfrac{p}{q})q) = (\tfrac{q}{p})^{X_n}(p+q) = (\tfrac{q}{p})^{X_n} = Z_n \ .\end{aligned}$$

Remark that, more generally, the product of independent r.v.'s having a mean equal to 1 is always a martingale, with respect to the natural filtration. Here we are dealing with an instance of this case, as the r.v.'s $(\frac{q}{p})^{Y_n}$, are actually independent and have mean equal to 1.

c) As $n \wedge \tau$ is a bounded stopping time, by the stopping theorem $\mathrm{E}[Z_{n\wedge\tau}] = \mathrm{E}[Z_n] = 1$. By a) we have $\tau < +\infty$. Therefore $\lim_{n\to\infty} Z_{n\wedge\tau} = Z_\tau$ a.s. As $\frac{q}{p} > 1$ and $-a \le X_{n\wedge\tau} \le b$, we have $(\frac{q}{p})^{-a} \le \lim_{n\to\infty} Z_{n\wedge\tau} \le (\frac{q}{p})^b$ and we can apply Lebesgue's theorem and derive that $\mathrm{E}[Z_\tau] = \lim_{n\to\infty}\mathrm{E}[Z_{n\wedge\tau}] = 1$.

d) As X_τ can take only the values $-a, b$,

$$1 = \mathrm{E}[Z_\tau] = \mathrm{E}[(\tfrac{q}{p})^{X_\tau}] = (\tfrac{q}{p})^b\mathrm{P}(X_\tau = b) + (\tfrac{q}{p})^{-a}\mathrm{P}(X_\tau = -a) \ .$$

As $\tau < +\infty$, $\mathrm{P}(X_\tau = -a) = 1 - \mathrm{P}(X_\tau = b)$ and therefore the previous relation gives

$$1 - (\tfrac{q}{p})^{-a} = \mathrm{P}(X_\tau = b)\big((\tfrac{q}{p})^b - (\tfrac{q}{p})^{-a}\big)$$

i.e.

$$\mathrm{P}(X_\tau = b) = \frac{1 - (\frac{q}{p})^{-a}}{(\frac{q}{p})^b - (\frac{q}{p})^{-a}}$$

and therefore

$$\mathrm{P}(X_\tau=-a)=\frac{(\frac{q}{p})^b-1}{(\frac{q}{p})^b-(\frac{q}{p})^{-a}}\cdot$$

Exercise 4. a) We have, recalling Remark 3.14,

$$\mathrm{E}[M_t\,|\,\mathscr{F}_s]=\mathrm{e}^{\lambda t}\mathrm{E}(B_t\,|\,\mathscr{F}_s)-\lambda\int_0^t \mathrm{e}^{\lambda u}\mathrm{E}[B_u\,|\,\mathscr{F}_s]\,du$$

Now

$$\int_0^t \mathrm{e}^{\lambda u}\mathrm{E}[B_u\,|\,\mathscr{F}_s]\,du=\int_0^s \mathrm{e}^{\lambda u}B_u\,du+\int_s^t \mathrm{e}^{\lambda u}B_s\,du$$

so that

$$\mathrm{E}([M_t\,|\,\mathscr{F}_s]=\mathrm{e}^{\lambda t}B_s-\lambda\int_0^s \mathrm{e}^{\lambda u}B_u\,du-(\mathrm{e}^{\lambda t}-\mathrm{e}^{\lambda s})B_s=M_s\ .$$

b) Let us write down the increments of M, trying to express them in terms of the increments of the Brownian motion. We have

$$\begin{aligned}M_t-M_s&=\mathrm{e}^{\lambda t}B_t-\mathrm{e}^{\lambda s}B_s-\lambda\int_s^t \mathrm{e}^{\lambda u}B_u\,du=\\&=\mathrm{e}^{\lambda t}(B_t-B_s)+(\mathrm{e}^{\lambda t}-\mathrm{e}^{\lambda s})B_s-\lambda\int_s^t \mathrm{e}^{\lambda u}(B_u-B_s)\,du-\lambda\int_s^t \mathrm{e}^{\lambda u}B_s\,du=\\&=\mathrm{e}^{\lambda t}(B_t-B_s)-\lambda\int_s^t \mathrm{e}^{\lambda u}(B_u-B_s)\,du\ .\end{aligned}$$

As M_t-M_s is a function of the increments of B after time s, it follows immediately that it is independent of $\mathscr{F}_s$ (recall Remark 2.9).