UNIVERSITÀ DI ROMA TOR VERGATA

Laurea Magistrale in Matematica Pura e Applicata

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Exercise 1 Let B be a Brownian motion. Compute

a) $\lim_{t\to+\infty} \mathrm{E}[1_{\{B_t\le a\}}]$

b) $\lim_{t\to+\infty} \mathrm{E}[B_t 1_{\{B_t\le a\}}]$

Exercise 2 Let B be a Brownian motion and let $b \in \mathbb{R}$ and $\sigma > 0$. Let $X_t = \mathrm{e}^{bt+\sigma B_t}$.

a) Investigate the existence and finiteness of the a.s. limit

$$\lim_{t\to+\infty} X_t$$

according to the possible values of b, σ for $b \neq 0$.

b) Investigate the existence and finiteness of

$$\lim_{t\to+\infty} \mathrm{E}[X_t] \tag{1}$$

according to the possible values of b, σ.

Exercise 3 Let B be a Brownian motion and let $b \in \mathbb{R}$ and $\sigma > 0$.

a) For which values of b, σ, $b \neq 0$ is the integral

$$\int_0^{+\infty} \mathrm{e}^{bu+\sigma B_u}\, du$$

a.s. finite?

b1) Prove that the r.v.

$$\int_0^1 1_{\{B_u>0\}}\, du$$

is > 0 a.s.

b2) Prove that

$$\lim_{t\to+\infty} \int_0^t 1_{\{B_u>0\}}\, du = +\infty \qquad a.s.$$

b3) Deduce that

$$\int_0^{+\infty} \mathrm{e}^{\sigma B_u}\, du = +\infty \qquad a.s.$$

c) For which values of b, σ is

$$\mathrm{E}\Big[\int_0^{+\infty} \mathrm{e}^{bu+\sigma B_u}\, du\Big] < +\infty\,?$$

Exercise 4 Let $B = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (B_t)_t, \mathrm{P})$ a Brownian motion. Compute

$$\mathrm{E}\Big(\int_s^t B_u^2\, du \,\big|\, \mathscr{F}_s\Big) \quad \text{and} \quad \mathrm{E}\Big(\int_s^t B_u^2\, du \mid B_s\Big)$$

Exercise 5 Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (B_t)_t, \mathrm{P})$ be a natural Brownian motion and, for $0 \le t \le 1$, let

$$X_t = B_t - t B_1$$

The process $(X_t)_t$, defined for $0 \le t \le 1$, is the *Brownian bridge*.

a) Show that $(X_t)_t$ is a centered Gaussian process independent of B_1. Compute $\mathrm{E}(X_t X_s)$.

b) Show that the r.v.

$$X_t - \frac{1-t}{1-s}\, X_s$$

is independent of X_s.

c) Compute $\mathrm{E}(X_t \mid X_s)$ and show that, denoting $\mathscr{G}_s = \sigma(X_u, u \le s)$, for $s \le t$

$$\mathrm{E}(X_t \mid \mathscr{G}_s) = \mathrm{E}(X_t \mid X_s)\ .$$

d) Compute $\mathrm{E}(X_t \mid \mathscr{F}_s)$. Do the σ-algebras $\mathscr{F}_s$ and $\mathscr{G}_s$ coincide?

e) Compute the finite dimensional distributions of $(X_t)_t$ $(0 \le t \le 1)$ and show that they coincide with the finite dimensional distributions of $(B_t)_t$ conditioned given $B_1 = 0$.

Solutions

Exercise 1. a) We have

$$\mathrm{E}[1_{\{B_t\le a\}}]=\mathrm{P}(B_t\le a)=\mathrm{P}(\sqrt{t}B_1\le a)=\mathrm{P}\Big(B_1\le \frac{a}{\sqrt{t}}\Big)$$

and therefore

$$\lim_{t\to+\infty}\mathrm{E}[1_{\{B_t\le a\}}]=\mathrm{P}(B_1\le 0)=\frac{1}{2}\,\cdot$$

b) We have

$$\mathrm{E}[B_t1_{\{B_t\le a\}}]=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^a x\mathrm{e}^{-\frac{x^2}{2t}}\,dx=-\frac{t}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{x^2}{2t}}\Big|_{-\infty}^a=-\frac{t}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{a^2}{2t}}$$

which, as $t\to-\infty$ converges to $-\infty$.

Exercise 2. a) By the iterated Logarithm Law we have $|B_t|\le(1+\varepsilon)\sqrt{2t\log\log t}$ for t large. Therefore

$$\lim_{t\to+\infty}X_t=0$$

if $b<0$, whatever the value of σ. The same arguments give $\lim_{t\to+\infty}X_t=+\infty$ if $b>0$ (again for every σ). If $b=0$ and $\sigma>0$ we know that, by the behavior of the Brownian motion, $\overline{\lim}_{t\to+\infty}X_t=+\infty$, $\underline{\lim}_{t\to+\infty}X_t=0$.

b) We have

$$\mathrm{E}[X_t]=\mathrm{e}^{bt}\mathrm{E}[\mathrm{e}^{\sigma B_t}]=\mathrm{e}^{(b+\frac{\sigma^2}{2})t}$$

so that the limit (1) is finite if and only if $b\le-\frac{\sigma^2}{2}$. Remark that in the range $b\in]-\frac{\sigma^2}{2},0[$ we have $\lim_{t\to+\infty}X_t=0$, but $\lim_{t\to+\infty}\mathrm{E}[X_t]=+\infty$.

Exercise 3. a) By the iterated Logarithm Law we have $|B_t|\le(1+\varepsilon)\sqrt{2t\log\log t}$ for t large. Hence if $b>0$ it is easy to see (this is also Exercise 2 a) that $\mathrm{e}^{bu+\sigma B_u}\to_{u\to+\infty}+\infty$, hence in this case the integrand itself diverges and the integral diverges. If $b<0$ conversely we have, for t large,

$$\mathrm{e}^{bu+\sigma B_u}\le\exp\big(-bt+(1+\varepsilon)\sqrt{2t\log\log t}\big)\le\mathrm{e}^{-bt/2}\,.$$

Hence the integral converges to a finite r.v.

b1) The integral can vanish only if the integrand, which is ≥ 0, vanishes a.s. We know however by the Iterated Logarithm Law that the Brownian path takes a.s. strictly positive values in every neighborhood of 0. As the paths are continuous they are therefore strictly positive on a set of times of strictly positive Lebesgue measure a.s.

b2) By a change of variable and using the scaling properties of the Brownian motion we have

$$\int_0^t 1_{\{B_u>0\}}\,du=t\int_0^1 1_{\{B_{tv}>0\}}\,dv=t\int_0^1 1_{\{\frac{1}{\sqrt{t}}B_{tv}>0\}}\,dv\sim t\int_0^1 1_{\{B_v>0\}}\,dv$$

as $v \mapsto \frac{1}{\sqrt{t}}\, B_{tv}$ is again a Brownian motion. Now we have

$$\lim_{t\to+\infty} t \int_0^1 1_{\{B_u>0\}}\, du = +\infty$$

as we have seen in b1) that the r.v. $\int_0^1 1_{\{B_u>0\}}\, du$ is strictly positive a.s. Hence, as the 2 r.v.'s $\int_0^t 1_{\{B_u>0\}}\, du$ and $t \int_0^1 1_{\{B_u>0\}}\, du$ have the same distribution for every t, we have

$$\lim_{t\to+\infty} \int_0^t 1_{\{B_u>0\}}\, du = +\infty \qquad \text{in probability} .$$

In order to prove the a.s. convergence, it suffices to remark that the limit

$$\lim_{t\to+\infty} \int_0^t 1_{\{B_u>0\}}\, du$$

exists a.s. as the integral is an increasing function of t. Hence we conclude because the a.s. limit and the limit in probability necessarily must coincide.

b3) Let $\sigma > 0$. It suffices to remark that

$$\mathrm{e}^{\sigma B_t} \ge 1_{\{B_t \ge 0\}}$$

and then to apply b2). If $\sigma < 0$, of course, just remark that $\sigma B_t = -\sigma(-B_t)$, where $(-B_t)_t$ is again a Brownian motion and now $-\sigma > 0$.

c) By Fubini's theorem and recalling the expression of the Laplace transform of the Gaussian distributions, we have

$$\mathrm{E}\Big[\int_0^{+\infty} \mathrm{e}^{bu+\sigma B_u}\, du\Big] = \int_0^{+\infty} \mathrm{e}^{(b+\frac{\sigma^2}{2})u}\, du \ .$$

Then it is clear that the expectation is finite if and only if $b < -\frac{\sigma^2}{2}$. The integral is actually easily computed giving, in conclusion

$$\mathrm{E}\Big[\int_0^{+\infty} \mathrm{e}^{bu+\sigma B_u}\, du\Big] = \begin{cases} \frac{1}{\frac{\sigma^2}{2}-b} & \text{if } b < -\frac{\sigma^2}{2} \\ +\infty & \text{otherwise.} \end{cases}$$

Exercise 4. The idea is always to split B_u into the sum of B_s and of the increment $B_u - B_s$. As $B_u^2 = (B_u - B_s + B_s)^2 = (B_u - B_s)^2 + B_s^2 + 2B_s(B_u - B_s)$, we have

$$\int_s^t B_u^2\, du = (t-s)B_s + \int_s^t (B_u - B_s)^2\, du + 2B_s \int_s^t (B_u - B_s)\, du$$

Now B_s is already $\mathscr{F}_s$-measurable whereas we have

$$\mathrm{E}\Big(\int_s^t (B_u - B_s)^2\,du \,\Big|\, \mathscr{F}_s\Big) = \int_s^t \mathrm{E}\big((B_u - B_s)^2 \,\big|\, \mathscr{F}_s\big)\,du = \int_s^t (u-s)\,du = \frac{1}{2}\,(t-s)^2$$

the r.v. $B_u - B_s$ being independent of $\mathscr{F}_s$. By the same argument

$$\mathrm{E}\Big(2B_s \int_s^t (B_u - B_s)\,du \,\Big|\, \mathscr{F}_s\Big) = 2B_s \int_s^t \mathrm{E}\big(B_u - B_s \,\big|\, \mathscr{F}_s\big)\,du = 0$$

so that finally

$$\mathrm{E}\Big(\int_s^t B_u^2\,du \,\Big|\, \mathscr{F}_s\Big) = (t-s)B_s + \frac{1}{2}\,(t-s)^2\;.$$

As this quantity is already $\sigma(B_s)$-measurable, we have

$$\mathrm{E}\Big(\int_s^t B_u^2\,du \mid B_s\Big) = \mathrm{E}\Big[\mathrm{E}\Big(\int_s^t B_u^2\,du \,\Big|\, \mathscr{F}_s\Big) \,\Big|\, B_s\Big] = (t-s)B_s + \frac{1}{2}\,(t-s)^2$$

Exercise 5. a) If $t_1, \dots, t_n \in \mathbb{R}^+$, then, as $(X_{t_1}, \dots, X_{t_n})$ is a linear function of the vector $(B_{t_1}, \dots, B_{t_n}, B_1)$, it is jointly Gaussian. Moreover, if $t \le 1$,

$$\mathrm{E}[X_t B_1] = \mathrm{E}[(B_t - tB_1)B_1] = t \wedge 1 - t = 0\;.$$

The two r.v.'s X_t and B_1, being jointly Gaussian and uncorrelated, are independent. X_t is centered and, if $t > s$,

$$\mathrm{E}[X_t X_s] = \mathrm{E}[(B_t - tB_1)(B_s - sB_1)] = s - st - st + st = s(1-t)\;.$$

b) As X_s and $X_t - \frac{1-t}{1-s}\,X_s$ are jointly Gaussian, we need only show that they are not correlated. We have

$$\mathrm{E}\big[\big(X_t - \tfrac{1-t}{1-s}\,X_s\big)X_s\big] = s(1-t) - \frac{1-t}{1-s}\,s(1-s) = 0\;.$$

c) We have

$$\mathrm{E}[X_t \mid X_s] = \mathrm{E}\big[X_t - \tfrac{1-t}{1-s}\,X_s + \tfrac{1-t}{1-s}\,X_s \,\big|\, X_s\big] = \frac{1-t}{1-s}\,X_s \tag{2}$$

as $X_t - \frac{1-t}{1-s}\,X_s$ is independent of X_s and centered. $\mathrm{E}(X_t \mid X_s) = \mathrm{E}(X_t \mid \mathscr{G}_s)$ follows if we show that $X_t - \frac{1-t}{1-s}\,X_s$ is independent of $\mathscr{G}_s$ and then repeat the argument of (2). In order to obtain this we know that it is sufficient to show that $X_t - \frac{1-t}{1-s}\, X_s$ is independent of X_u for every $u \le s$. This is true as

$$\mathrm{E}\big[\big(X_t - \tfrac{1-t}{1-s}\, X_s\big)X_u\big] = u(1-t) - \frac{1-t}{1-s}u(1-s) = 0\ .$$

d) We have, for $s \le t \le 1$,

$$\mathrm{E}[X_t \,|\, \mathscr{F}_s] = \mathrm{E}[B_t - tB_1 \,|\, \mathscr{F}_s] = B_s - tB_s = (1-t)B_s\ .$$

This result is different from

$$\mathrm{E}[X_t \,|\, \mathscr{G}_s] = \frac{1-t}{1-s}\, X_s$$

obtained in c), as it is easy to see that $\mathrm{Var}(\frac{1-t}{1-s}\, X_s) = (1-t)^2\, \frac{s}{1-s}$ whereas $\mathrm{Var}((1-t)B_s) = (1-t)^2 s$. Therefore the two σ-algebras $\mathscr{F}_s$ and $\mathscr{G}_s$ are different.

e) Let $0 \le t_1 < \dots < t_m \le 1$. Then the conditional law of the vector $(B_{t_1}, \dots, B_{t_m})$ given $B_1 = 0$ is Gaussian and can be computed as explained in §3.4. It has mean 0, as in (3.17) the quantities m_X, m_Y and y vanish. The covariance matrix is given by (3.18), where we identify

$$\begin{aligned} C_X &= \text{covariance matrix of } (B_{t_1}, \dots, B_{t_m})\\ C_{X,Y} &= \text{vector of the covariances of } B_{t_i} \text{ and } B_1\\ C_Y &= \text{covariance matrix of } B_1\ . \end{aligned}$$

Therefore, denoting $K = (k_{ij})_{ij}$ the covariance matrix to be computed, we find

$$k_{ij} = t_i \wedge t_j - t_i t_j$$

that coincides with the covariance matrix of $(X_{t_1}, \dots, X_{t_m})$, as obtained in the second part of a).