UNIVERSITÀ DI ROMA TOR VERGATA

Laurea Magistrale in Matematica Pura e Applicata

Corso di *EP-Elementi di Probabilità*
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Exercise 1 Let (B_1, B_2, B_3) be a 3-dimensional Brownian motion and

$$\begin{aligned} X_t &= \int_0^t \sin(B_3(s))\, dB_1(s) + \int_0^t \cos(B_3(s))\, dB_2(s) \\ Y_t &= \int_0^t \cos(B_3(s))\, dB_1(s) + \int_0^t \sin(B_3(s))\, dB_2(s)\ . \end{aligned} \tag{1}$$

a) Prove that $(X_t)_t$ and $(Y_t)_t$ are Brownian motions.
b1) Compute, for $s, t > 0$, $\mathrm{E}[X_t Y_s]$.
b2) Is $(X_t, Y_t)_t$ a 2-dimensional Brownian motion?
c) Assume instead that

$$Y_t = -\int_0^t \cos(B_3(s))\, dB_1(s) + \int_0^t \sin(B_3(s))\, dB_2(s)\ .$$

Is now $(X_t, Y_t)_t$ a 2-dimensional Brownian motion?

Exercise 2 Let (B_1, B_2) a two-dimensional Brownian motion and let us consider the two processes

$$\begin{aligned} dS_1(t) &= r_1 S_1(t)\, dt + \sigma_1 S_1(t)\, dB_1(t) \\ dS_2(t) &= r_2 S_2(t)\, dt + \sigma_2 S_2(t)\, dB_2(t) \end{aligned}$$

where $r_1, r_2, \sigma_1, \sigma_2$ are real numbers and with the initial conditions $S_0 = s_0$, $S_1 = s_1$.

a) Prove that the process $X_t = S_1(t) S_2(t)$ is the solution of a stochastic differential equation with respect to a new linear Brownian motion to be determined.

b1) Answer the same questions as in a) if we had, for $-1 \le \rho \le 1$,

$$\begin{aligned} dS_1(t) &= r_1 S_1(t)\, dt + \sigma_1 S_1(t)\, dB_1(t) \\ dS_2(t) &= r_2 S_2(t)\, dt + \sigma_2 \sqrt{1-\rho^2}\, S_2(t)\, dB_2(t) + \sigma_2 \rho S_2(t)\, dB_1(t) \end{aligned}$$

b2) Compute the expectation of $X_t = S_1(t) S_2(t)$ and also $\mathrm{E}[X_t^2]$. For which values of ρ each of these quantities is maximum?

b3) Which is the generator of (S_1, S_2)?

Exercise 3 Let us consider the equation

$$d\xi_t = -\gamma\Big(\int_0^t \xi_s\,ds\Big)\,dt + \sigma\,dB_t\ ,\qquad \xi_0 = x \tag{2}$$

where σ, γ are real numbers, $\sigma > 0$. In some sense this is a stochastic equation whose drift coefficient does not depend only on the position of the process at time t but also on its past.

a) Let us consider the 2-dimensional SDE

$$\begin{aligned} d\zeta_t &= -\gamma\eta_t\,dt + \sigma\,dB_t \\ d\eta_t &= \zeta_t\,dt \end{aligned} \tag{3}$$

with the initial conditions $\zeta_0 = x$, $\eta_0 = 0$. Check carefully that if $(\zeta_t, \eta_t)_t$ is a solution of (3) then ζ is a solution of (2) and prove that for equation (2) existence and uniqueness of the solution hold.

b) Prove that ξ is a Gaussian process.

c) Assume $\gamma = 1$. Which is the value of $\mathrm{E}[\xi_t]$?

Solutions

Exercise 1. a) We can write

$$X_t = \int_0^t Z_s \, dB_s$$

with $Z_s = (\sin(B_3(s)), \cos(B_3(s)))$. As $|Z_s| = 1$ for every s, X is a Brownian motion by Corollary 7.24. Same argument for Y.

b1) Let us assume $s \le t$. As $\langle B_1, B_2\rangle_t \equiv 0$, we have

$$\begin{aligned}
&\mathrm{E}[X_t Y_s] = \\
&= \mathrm{E}\Big[\int_0^t \sin(B_3(u))\, dB_1(u) \int_0^s \cos(B_3(v))\, dB_1(v) + \\
&\qquad + \int_0^t \cos(B_3(u))\, dB_2(u) \int_0^s \sin(B_3(v))\, dB_2(v)\Big] = \\
&= \mathrm{E}\Big[\int_0^s \sin(B_3(u))\, dB_1(u) \int_0^s \cos(B_3(v))\, dB_1(v) + \\
&\qquad + \int_0^s \cos(B_3(u))\, dB_2(u) \int_0^s \sin(B_3(v))\, dB_2(v)\Big] = \\
&= \mathrm{E}\Big[\int_0^s \sin(B_3(u)) \cos(B_3(u))\, du + \int_0^s \cos(B_3(u)) \sin(B_3(u))\, du\Big] = \\
&= \int_0^s \mathrm{E}[\sin(2B_3(u))]\, du = 0
\end{aligned}$$

as the r.v. $\sin(2B_3(s))$ has the same distribution as $\sin(-2B_3(s)) = -\sin(2B_3(s))$, and has therefore mathematical expectation that is equal to 0.

b2) We have easily

$$d\langle X, Y\rangle_t = 2\sin(B_3(t))\cos(B_3(t))\, dt \not\equiv 0 \ .$$

hence $t \mapsto (X_t, Y_t)$ cannot be a Brownian motion.

c) With the new definition we can write

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \int_0^t O_s \, dB_s$$

where we denote by B the 2-dimensional Brownian motion (B_1, B_2) and

$$O_s = \begin{pmatrix} \sin(B_3(s)) & \cos(B_3(s)) \\ -\cos(B_3(s)) & \sin(B_3(s)) \end{pmatrix} \ .$$

It is immediate that $s \mapsto O_s$ is an orthogonal-matrix-valued process. Hence the required statement follows from Proposition 7.26 of the class notes.

Exercise 2. a) The clever reader has certainly remarked that the two processes S_1, S_2 are geometric Brownian motions for which an explicit solution is known, which allows to come correctly to the right answer. Let us however work otherwise. Remark that the associated increasing process $\langle S_1, S_2\rangle_t = \sigma_1\sigma_2 S_1(t)S_2(t)\langle B_1, B_2\rangle_t$ vanishes so that

$$\begin{aligned} dS_1(t)S_2(t) &= S_1(t)\,dS_2(t) + S_2(t)\,dS_1(t) = \\ &= S_1(t)\big(r_2S_2(t)\,dt + \sigma_2S_2(t)\,dB_2(t)\big) + S_2(t)\big(r_1S_1(t)\,dt + \sigma_1S_1(t)\,dB_1(t)\big) \end{aligned}$$

so that, if $X_t = S_1(t)S_2(t)$,

$$dX_t = (r_1 + r_2)X_t\,dt + X_t\big(\sigma_1\,dB_1(t) + \sigma_2\,dB_2(t)\big)$$

Now, if we define

$$\widetilde{B}_t = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\,\big(\sigma_1 B_1(t) + \sigma_2 B_2(t)\big)$$

this is a Brownian motion and the above relation for dX_t becomes

$$dX_t = (r_1 + r_2)X_t\,dt + \sqrt{\sigma_1^2 + \sigma_2^2}\,X_t\,d\widetilde{B}_t$$

X is therefore also a geometric Brownian motion and

$$X_t = x_0 e^{\left(r_1+r_2-\frac{1}{2}\,(\sigma_1^2+\sigma_1^2)\right)+\sqrt{\sigma_1^2+\sigma_2^2}\widetilde{B}_t} \tag{4}$$

b1) In order to simplify the formulas let us denote $\overline{\rho} = \sqrt{1-\rho^2}$. We can repeat the previous arguments, but now $\langle S_1, S_2\rangle_t = \sigma_1\sigma_2 S_1(t)S_2(t)\langle B_1, \rho B_1 + \overline{\rho}B_2\rangle_t = \sigma_1\sigma_2 S_1(t)S_2(t)\rho\,dt$. Therefore

$$\begin{aligned} dS_1(t)S_2(t) &= S_1(t)\,dS_2(t) + S_2(t)\,dS_1(t) + \sigma_1\sigma_2 S_1(t)S_2(t)\rho\,dt = \\ &= S_1(t)\big(r_2S_2(t)\,dt + \sigma_2\overline{\rho}\,S_2(t)\,dB_2(t) + \sigma_2\rho S_2(t)\,dB_1(t)\big) + \\ &\qquad + S_2(t)\big(r_1S_1(t)\,dt + \sigma_1S_1(t)\,dB_1(t)\big) + \sigma_1\sigma_2S_1(t)S_2(t)\rho\,dt \end{aligned}$$

so that, as $X_t = S_1(t)S_2(t)$,

$$dX_t = (r_1 + r_2 + \sigma_1\sigma_2\rho)X_t\,dt + X_t\big((\sigma_1 + \rho\sigma_2)\,dB_1(t) + \sigma_2\overline{\rho}\,dB_2(t)\big)$$

Now, as $(\sigma_1 + \rho\sigma_2)^2 + (\sigma_2\overline{\rho})^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$ we have a new Brownian motion by letting

$$\widetilde{B}_t = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}\,\big((\sigma_1 + \rho\sigma_2)\,B_1(t) + \sigma_2\overline{\rho}\,B_2(t)\big)$$

and we find again that X is a geometric Brownian motion with stochastic differential

$$dX_t = (r_1 + r_2 + \sigma_1\sigma_2\rho)X_t\,dt + \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\,X_t\,d\widetilde{B}_t\ .$$

b2) We know that $X_t = S_1(t)S_2(t)$ is a geometric Brownian motion and that, setting $\sigma_\rho = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$, we have

$$X_t = x_0 \mathrm{e}^{(r_1+r_2+\sigma_1\sigma_2\rho-\frac{1}{2}\,\sigma_\rho^2)t+\sigma_\rho B_t}$$

Recalling the expression of the Laplace transforms of the Gaussian distributions

$$\mathrm{E}[X_t] = x_0 \mathrm{e}^{(r_1+r_2+\sigma_1\sigma_2\rho-\frac{1}{2}\,\sigma_\rho^2)t}\mathrm{e}^{\frac{1}{2}\,\sigma_\rho^2} = x_0\mathrm{e}^{(r_1+r_2+\sigma_1\sigma_2\rho)t}$$

which is maximum for $\rho = 1$ if $\sigma_1\sigma_2 > 0$ and for $\rho = -1$ otherwise. Similarly we have

$$\mathrm{E}[X_t^2] = x_0^2\mathrm{e}^{2(r_1+r_2+\sigma_1\sigma_2\rho)t-\sigma_\rho^2 t}\mathrm{e}^{2\sigma_\rho^2 t} = x_0^2\mathrm{e}^{2(r_1+r_2+\sigma_1\sigma_2\rho)t+\sigma_\rho^2 t} = x_0^2\mathrm{e}^{(2(r_1+r_2)+\sigma_1^2+\sigma_2^2+4\sigma_1\sigma_2\rho)t}$$

which is maximum for $\rho = 1$ or $\rho = -1$ as above.

b3) The process $t \mapsto (S_1(t), S_2(t))$ has diffusion matrix

$$\sigma(x) = \begin{pmatrix} \sigma_1 x_1 & 0 \\ \sigma_2\rho x_2 & \sigma_2\overline{\rho}x_2 \end{pmatrix}\ .$$

Therefore

$$\sigma(x)\sigma(x)^* = \begin{pmatrix} \sigma_1 x_1 & 0 \\ \sigma_2\rho x_2 & \sigma_2\overline{\rho}x_2 \end{pmatrix}\begin{pmatrix} \sigma_1 x_1 & \sigma_2\rho x_2 \\ 0 & \sigma_2\overline{\rho}x_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 x_1^2 & \sigma_1\sigma_2\rho x_1 x_2 \\ \sigma_1\sigma_2\rho x_1 x_2 & \sigma_2^2 x_2^2 \end{pmatrix}$$

and therefore the requested generator is

$$L = \frac{1}{2}\Big(\sigma_1^2 x_1^2 \frac{\partial^2}{\partial x_1^2} + \sigma_2^2 x_2^2 \frac{\partial^2}{\partial x_2^2} + 2\sigma_1\sigma_2\rho x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2}\Big) + r_1 x_1 \frac{\partial}{\partial x_1} + r_2 x_2 \frac{\partial}{\partial x_2}\ \cdot$$

Exercise 3. a) If $(\zeta_t, \eta_t)_t$ is a solution of (3) then clearly $\eta_t = \int_0^t \zeta_s\,ds$ and replacing in the first equation we have

$$d\zeta_t = -\gamma\Big(\int_0^t \zeta_s\,ds\Big)\,dt + \sigma\,dB_t$$

i.e. ζ is a solution of (2). This proves existence of the solutions for (2). Conversely if ξ is a solution of (2), then it is immediate that if $\eta_t = \int_0^t \xi_s\,ds$ then $(\xi_t, \eta_t)_t$ is a solution of (3) and this proves uniqueness.

b) Remark that (3) can be written

$$d\begin{pmatrix}\zeta_t\\ \eta_t\end{pmatrix}=\begin{pmatrix}0 & -\gamma\\ 1 & 0\end{pmatrix}dt+\begin{pmatrix}\sigma\\ 0\end{pmatrix}dB_t$$

This is a SDE with a drift that is linear in the state variable and a diffusion coefficient that is constant, i.e. a multidimensional Ornstein-Uhlenbeck process which we know to be Gaussian.

c) From the formulas of the Ornstein-Uhlenbeck processes we know that the process $\begin{pmatrix}\zeta_t\\ \eta_t\end{pmatrix}$ has mean

$$\mathrm{e}^{tG}\begin{pmatrix}x\\ 0\end{pmatrix} \tag{5}$$

where G is the matrix

$$G=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

Let us compute e^{tG}. Taking the powers of G we have

$$G^2=-I,\quad G^3=-G,\quad G^4=I,\dots$$

i.e.

$$G^{2m}=(-1)^m I,\quad G^{2m+1}=(-1)^m G$$

Therefore

$$\mathrm{e}^{tG}=\sum_{n=0}^{\infty}\frac{1}{n!}(tG)^n=I\sum_{m=0}^{\infty}\frac{1}{(2m)!}(-1)^m t^{2m}+G\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}(-1)^m t^{2m+1}=$$
$$=I\cos t+G\sin t=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}.$$

This and (5) give

$$\mathrm{E}[\xi_t]=x\cos t$$

and, by the way, $\mathrm{E}[\eta_t]=x\sin t$, which was not required. In particular the vector of the 2 expectations $(\mathrm{E}[\xi_t],\mathrm{E}[\eta_t])$ uniformly moves on the circle of radius x.

The computation of the exponential of tG would be simplified by the remark that G is antisymmetric so that e^{tG} is orthogonal. The computation above can be performed also for $\gamma\neq 1$ with a few additional difficulties.