UNIVERSITÀ DI ROMA TOR VERGATA

Laurea Magistrale in Matematica Pura e Applicata

Corso di *EP-Elementi di Probabilità*
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Esonero 1 del 22 aprile 2016

Exercise 1 Let $B = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (B_t)_t, \mathrm{P})$ be a Brownian motion.

a) Prove that there exists $\alpha \in \mathbb{R}$ such that

$$X_t = B_t^4 - 6tB_t^2 + \alpha t^2$$

is a martingale. (Recall: $(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$, $\mathrm{E}[Z^4] = 3$ if $Z \sim N(0,1)$).

b) Let $a > 0$ and let τ the exit time of B from $]-a, a[$. Recall that we know already that τ is integrable and that $\mathrm{E}[\tau] = a^2$. Compute $\mathrm{E}[\tau^2]$ and $\mathrm{Var}(\tau)$.

Exercise 2 Let $B = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (B_t)_t, \mathrm{P})$ be a Brownian motion. Recall that, for every $\lambda \in \mathbb{R}$,

$$M_t = \mathrm{e}^{\lambda B_t - \frac{1}{2}\lambda^2 t}$$

is a positive martingale with expectation equal to 1. For $T > 0$ let us define on $(\Omega, \mathscr{F})$ a probability Q by, for every $A \in \mathscr{F}$,

$$\mathrm{Q}(A) = \mathrm{E}[M_T 1_A] = \int_A M_T \, d\mathrm{P} \; .$$

Let us denote E^{Q} the expectation with respect to Q.

a) Prove that if $t \le T$ and $A \in \mathscr{F}_t$

$$\mathrm{Q}(A) = \mathrm{E}[M_t 1_A] \; .$$

b1) Prove that, for every $z \in \mathbb{C}$,

$$Z_t = \mathrm{e}^{zB_t - \frac{1}{2}z^2 t}$$

is a complex martingale.

b2) Compute $\mathrm{E}^{\mathrm{Q}}[\mathrm{e}^{i\theta B_t}]$, for $\theta \in \mathbb{R}$. Is B a Brownian motion also with respect to Q?

b3) Prove that, with respect to Q, $X_t = B_t - \lambda t$ is a Brownian motion for $t \in [0, T]$.

Exercise 3 Let $B = (\Omega, \mathscr{F}, (\mathscr{F}_t)_t, (B_t)_t, \mathrm{P})$ be a $m$dimensional Brownian motion and

$$X_t = B_{g(t)} \; .$$

where $g(t) = 1 - \mathrm{e}^{-t}$.

a) Prove that X is a Markov process and compute its transition function. Is it time homogeneous?

b) Let $(\Omega, \mathscr{F}, (\mathscr{F}_s)_s, (X_t)_t, (\mathrm{P}^{x,s})_{x,s})$ a realization of the Markov process associated to the transition function computed in a). Prove that, for every starting point x and initial time s, $(X_t)_t$ converges as $t \to +\infty$ to a limit distribution to be determined, possibly depending on x, s.

c) Compute, for every $f \in \mathscr{C}_K^2(\mathbb{R}^m)$, the generator $A_s f(x)$.

Solutions

Exercise 1. a) As the expectation of a martingale is constant, in order for M to be a martingale we must have

$$\mathrm{E}[B_t^4] + \alpha t^2 = 6t\mathrm{E}[B_t^2]\ .$$

As $B_t^4 \sim t^2 Z^4$ with $Z \sim N(0, 1)$, the previous relation becomes $3t^2 + \alpha t^2 = 6t^2$, hence $\alpha = 3$. Let us prove that with this choice of α X is actually a martingale. We have, for $s \le t$,

$$\begin{aligned}
\mathrm{E}[B_t^4 \,|\, \mathscr{F}_s] &= \mathrm{E}[(B_s + (B_t - B_s))^4 \mathscr{F}_s] = \\
&= B_s^4 + 4\mathrm{E}[B_s^3(B_t - B_s)\mathscr{F}_s] + 6\mathrm{E}[B_s^2(B_t - B_s)^2\mathscr{F}_s] + 4\mathrm{E}[B_s(B_t - B_s)^3\mathscr{F}_s] + |E[(B_t - B_s)^4\mathscr{F}_s] = \\
&= B_s^4 + 4B_s^3\mathrm{E}[(B_t - B_s)\mathscr{F}_s] + 6B_s^2\mathrm{E}[(B_t - B_s)^2\mathscr{F}_s] + 4B_s\mathrm{E}[(B_t - B_s)^3\mathscr{F}_s] + |E[(B_t - B_s)^4\mathscr{F}_s] = \\
&= B_s^4 + 4B_s^3\mathrm{E}[(B_t - B_s)] + 6B_s^2\mathrm{E}[(B_t - B_s)^2] + 4B_s\mathrm{E}[(B_t - B_s)^3] + |E[(B_t - B_s)^4] = \\
&= B_s^4 + 6B_s^2(t - s) + 3(t - s)^2
\end{aligned}$$

and also

$$\begin{aligned}
\mathrm{E}[tB_t^2\mathscr{F}_s] &= t\mathrm{E}[(B_s + (B_t - B_s)^2\mathscr{F}_s] = t\big(B_s^2 + 2\mathrm{E}[B_s(B_t - B_s)\mathscr{F}_s] + \mathrm{E}[(B_t - B_s)^2\mathscr{F}_s]\big) = \\
&= t\big(B_s^2 + 2B_s\mathrm{E}[(B_t - B_s)\mathscr{F}_s] + \mathrm{E}[(B_t - B_s)^2]) = t(B_s^2 + (t - s)\big)\ .
\end{aligned}$$

Therefore

$$\mathrm{E}[X_t \,|\, \mathscr{F}_s] = B_s^4 + 6B_s^2(t - s) + 3(t - s)^2 - 6t(B_s^2 + (t - s)) + 3t^2 = B_s^4 - 6sB_s^2 + 3s^2 = X_s\ .$$

b) For every $t > 0$ we have, thanks to the stopping theorem,

$$0 = \mathrm{E}[X_{t\wedge\tau}] = \mathrm{E}[B_{t\wedge\tau}^4] - 6\mathrm{E}[(t\wedge\tau)B_{t\wedge\tau}^2] + 3\mathrm{E}[(t\wedge\tau)^2]\ . \tag{1}$$

We have

$$\lim_{t\to+\infty} \mathrm{E}[(t\wedge\tau)^2] = \mathrm{E}[\tau^2]$$

by Beppo Levi's theorem. Also, by Lebesgue's theorem,

$$\begin{aligned}
&\lim_{t\to+\infty} \mathrm{E}[B_{t\wedge\tau}^4] = \mathrm{E}[B_\tau^4] = a^4 \\
&\lim_{t\to+\infty} \mathrm{E}[(t\wedge\tau)B_{t\wedge\tau}^2] = \mathrm{E}[\tau B_\tau^2] = a^2\mathrm{E}[\tau] = a^4\ .
\end{aligned}$$

Actually we have $|B_{t\wedge\tau}| \le a$, $(t\wedge\tau)B_{t\wedge\tau}^2 \le \tau a$ and we know already that τ is integrable.

Therefore, taking the limit as $t \to +\infty$ (1) gives

$$0 = a^4 - 6a^4 + 3\mathrm{E}[\tau^2]$$

i.e.

$$\mathrm{E}[\tau^2] = \frac{5}{3}\,a^4$$

and $\mathrm{Var}(\tau) = \mathrm{E}[\tau^2] - \mathrm{E}[\tau]^2 = \frac{5}{3}\,a^4 - a^4 = \frac{2}{3}\,a^4$.

Exercise 2. a) We have

$$\mathrm{Q}(A) = \mathrm{E}[M_T 1_A] = \mathrm{E}\big[\mathrm{E}[M_T 1_A \,|\, \mathscr{F}_t]\big] = \mathrm{E}\big[1_A \mathrm{E}[M_T \,|\, \mathscr{F}_t]\big] = \mathrm{E}[M_t 1_A\ .]$$

b) We have

$$\mathrm{E}^{\mathrm{Q}}[\mathrm{e}^{i\theta B_t}] = \mathrm{E}[M_t \mathrm{e}^{i\theta B_t}] = \mathrm{e}^{-\frac{1}{2}\,\lambda^2 t}\mathrm{E}\Big[\mathrm{e}^{(\lambda+i\theta)B_t}\Big] = \mathrm{e}^{-\frac{1}{2}\,(\lambda^2-(\lambda+i\theta)^2)t} = \mathrm{e}^{-\frac{1}{2}\,t\theta^2}\mathrm{e}^{-i\lambda\theta t}\ .$$

We recognize the characteristic function of a $N(\lambda t, t)$ distributed r.v. Therefore B is not a Brownian motion with respect to Q (B_t is not centered).

c1) Very much similar to the case $z \in \mathbb{R}$ that we have seen in class: we have, for $s \le t$,

$$\begin{aligned}\mathrm{E}[Z_t \,|\, \mathscr{F}_s] &= \mathrm{e}^{-\frac{1}{2}\,z^2 t}\mathrm{E}[\mathrm{e}^{z(B_s+(B_t-B_s))} \,|\, \mathscr{F}_s] = \mathrm{e}^{zB_s-\frac{1}{2}\,z^2 t}\mathrm{E}[\mathrm{e}^{z(B_t-B_s)} \,|\, \mathscr{F}_s] = \\ &= \mathrm{e}^{zB_s-\frac{1}{2}\,z^2 t}\mathrm{E}[\mathrm{e}^{z(B_t-B_s)}] = \mathrm{e}^{zB_s-\frac{1}{2}\,z^2 t+\frac{1}{2}z^2(t-s)} = Z_s\ .\end{aligned}$$

c2) The simplest way of checking that $X_t = B_t - \lambda t$ is a Brownian motion is (recall Theorem 4.34) to verify that

$$Y_t = \mathrm{e}^{i\theta X_t+\frac{1}{2}\,\theta^2 t} = \mathrm{e}^{i\theta(B_t-\lambda t)+\frac{1}{2}\,\theta^2 t}$$

is a Q-martingale for every $\theta \in \mathbb{R}$. If $s \le t \le T$ and $A \in \mathscr{F}_s$, we have, again using a) and c1) for $z = \lambda + i\theta$,

$$\begin{aligned}\mathrm{E}^{\mathrm{Q}}[Y_t 1_A] &= \mathrm{E}[Y_t 1_A M_t] = \\ &= \mathrm{E}[\mathrm{e}^{i\theta(B_t-\lambda t)+\frac{1}{2}\,\theta^2 t}\mathrm{e}^{\lambda B_t-\frac{1}{2}\,\lambda^2 t}1_A] = \mathrm{E}[\mathrm{e}^{(\lambda+i\theta)B_t-\frac{1}{2}\,t(\lambda+i\theta)^2}1_A] = \\ &= \mathrm{E}[\mathrm{e}^{(\lambda+i\theta)B_s-\frac{1}{2}\,s(\lambda+i\theta)^2}1_A] = \mathrm{E}[Y_s M_s 1_A] = \mathrm{E}[Y_s M_T 1_A] = \mathrm{E}^{\mathrm{Q}}[Y_s 1_A]\ .\end{aligned}$$

Exercise 3. a) There are (at least) two possible ways of reasoning. First remark that X is clearly a Gaussian process for which we know a Markovianity criterion. Let us compute first the covariance function of X: as g is an increasing function, for $s \le t$,

$$K_{s,t} = \mathrm{Cov}(X_s, X_t) = \mathrm{E}[B_{g(s)} B_g(t)] = g(s) \wedge g(t) = g(s)\ .$$

Therefore we have, for $u \le s \le t$,

$$K_{u,s} K_{s,s}^{-1} K_{s,t} = g(u)\,\frac{1}{g(s)}\,g(s) = g(u) = K_{u,t}$$

which ensures that X is a Markov process *with respect to its natural filtration*. In order to determine the transition function $p(s, t, x, dy)$, let us recall that this is simply the conditional distribution of X_t given $X_s = x$. With the well-known formulas of the conditional distributions of jointly Gaussian r.v.'s $p(s,t,x,dy)$is Gaussian with mean

$$\mathrm{E}[X_t] + \frac{K_{s,t}}{K_{s,s}}\,(x - \mathrm{E}[X_s]) = x$$

and variance

$$K_{t,t} - \frac{K_{s,t}^2}{K_{s,s}} = g(t) - g(s) = \mathrm{e}^{-s} - \mathrm{e}^{-t}$$

i.e. $p(s,t,x,dy) \sim N(x, \mathrm{e}^{-s} - \mathrm{e}^{-t})$.

Second method. We can here check directly the Markov property with respect to the filtration $(\mathscr{F}_{g(t)})_t$. Actually, thanks to the freezing lemma and with the trick of decomposing into sum of the actual position and of the increment, we have for every bounded measurable function f

$$\mathrm{E}\big[f(B_{g(t)})\,|\,\mathscr{F}_{g(s)}\big] = \mathrm{E}\big[f(B_{g(t)} - B_{g(s)} + B_{g(s)})\,|\,\mathscr{F}_{g(s)}\big] = \Phi(B_{g(s)}) \tag{2}$$

where

$$\Phi(x) = \mathrm{E}\big[f(\underbrace{B_{g(t)} - B_{g(s)} + x})\big]\ .$$

Therefore (2) proves that X is a Markov process *with respect to its natural filtration* $(\mathscr{F}_{g(t)})_t$ and, remarking that the r.v. indicated by the down brace is $N(x, g(t) - g(s))$-distributed, this proves also that this is the transitions function.

b) The law of X_t starting at (s,x) is $p(s,t,x,dy) \sim N(x, \mathrm{e}^{-s} - \mathrm{e}^{-t})$, which, as $t \to +\infty$, converges to a $N(x, \mathrm{e}^{-s})$ distribution.

c) If $f \in \mathscr{C}_K^2(\mathbb{R}^m)$, the value of $A_s f(x)$ is given by

$$\lim_{h\to 0} \frac{1}{h}\,\big(T_{s,s+h} f(x) - f(x)\big)$$

(provided the limit exists) where

$$T_{s,s+h} f(x) = \int_{\mathbb{R}^m} f(y)\, p(s, s+h, x, dy)\ .$$

The crucial remark is that $p(s, s+h, x, dy)$ is the distribution of $x + B_{g(s+h)} - B_{g(s)} \sim x + B_{g(s+h)-g(s)}$, so that

$$\begin{aligned}A_s f(x) &= \lim_{h\to 0} \frac{1}{h}\,\big(\mathrm{E}[f(x + B_{g(s+h)-g(s)}) - f(x)\big) = \\ &= \lim_{h\to 0} \frac{1}{h}\,\frac{g(s+h) - g(s)}{g(s+h) - g(s)}\,\big(\mathrm{E}[f(x + B_{g(s+h)-g(s)}) - f(x)\big)\ .\end{aligned}$$

As

$$\lim_{h\to 0} \frac{g(s+h) - g(s)}{h} = \lim_{h\to 0} \frac{1 - \mathrm{e}^{-(s+h)} - 1 + \mathrm{e}^{-s}}{h} = \mathrm{e}^{-s} \lim_{h\to 0} \frac{1 - \mathrm{e}^{-h}}{h} = \mathrm{e}^{-s}$$

we have, going back to the generator of the Brownian motion that we already know,

$$A_s f(x) = \mathrm{e}^{-s} \lim_{h\to 0} \frac{1}{g(s+h)-g(s)} \big(\mathrm{E}[f(x+B_{g(s+h)-g(s)}) - f(x)\big) =$$
$$= \mathrm{e}^{-s} \lim_{t\to 0} \frac{1}{t} \big(\mathrm{E}[f(x+B_t) - f(x)\big) = \frac{\mathrm{e}^{-s}}{2} \triangle f(x)\ .$$