INTEGRAL GEOMETRY, HARMONIC ANALYSIS, POTENTIAL THEORY AND STATISTICAL MECHANICS ON NETWORKS

MASSIMO A. PICARDELLO (P.I.), FRANCESCO FIDALEO, AND FLORIN RADULESCU

1. ABSTRACT

This project deals with infinite networks, but an important part is devoted to trees T. What is unique to this special environment is the existence of a large group of isometries: the action of this group gives rise to convolutions and harmonic analysis. So, the first research line is aimed to study harmonic analysis on trees and representation theory of their group of automorphisms via a geometric approach. More than 35 years ago the P.I., jointly with A. Figa'-Talamanca, investigated spherical representation theory of simply transitive subgroups of Aut T, like free groups, by techniques based on analysis of Poisson kernels. The free group is a co-compact subgroup of Aut T, hence relative large. But the revolutionary idea here, in the spirit of Gelfand and helgason, is to look at functions on horospheres instead of vertices. The automorphism group Aut H of the horospherical fiber bundle H is much larger than Aut T, for instance it contains a "Cartan" subgroup A of parallel shifts along the fibers. The Poisson kernel and its powers lift to functions on H, that happen to be characters of A, and spherical representations are induced in the sense of Mackey from these characters.This leads to a detailed study of spherical representations of Aut H that extend the known representations of free groups. But at the same time we can consider horospheres of edges, or of oriented edges ("flags"), instead of vertices: we obtain a new theory of representations for each type of simplex in the simplicial complex T. This will open the way to extensions to spherical representations of higher rank buildings. At the same time, this allows us to study two new subjects: Radon transforms on T and eigenspaces of its Laplace operator. We shall obtain inversion formulas for all the Radon transforms (on vertices, edges and flags), and as a consequence the corresponding Plancherel measures. The tool is the spectral theory of the Laplacians, that will be computed by using the recurrence relations of its powers (hence the convolution product induced by the simply transitive subgroup. i.e. harmonic analysis and Gelfand pairs).

If homogeneous trees are a typical example of Bruhat-Tits buildings of rank 1, also semi-homogeneous trees are combinatorial rank 1 buildings. But in this context there are two dramatic differences. The first is that there is no transitive group of automorphisms on vertices, hence no step-one recurrence relation (no step 2 difference equations for spherical functions): however, there is a simply transitive subgroup on edges. The second is that the extremal eigenfunctios of the Laplacian are not powers of the Poisson kernel, hence do not lift to the horospherical bundles and cannot be computed via integral geometry: we shall need tools from potential theory (the generating functions of the Markov chain induced by the Laplacians). Nice Gelfand pairs will not exist any longer for the Laplace operator, although they will for its square, because there is a step-2 transitive subgroup of Aut T: but

the set of vertices at even distance is not a tree any longer (it is a network, more precisely a polygonal graph, originally studied by the P.I. and A. Iozzi 35 years ago for different goals). Still, we can study spectral theory of the Laplacian, and this will be a completely new theory, with very different spectra and Lp spherical functions.

On the other hand, using Markov chains to analyze eigenfunctions of the Laplacian leads us to a new insight: boundary theory for normalized eigenfunctions. The Poisson boundary representation is known for eigenfunctions of the Laplacian (see results of the P.I. and A. Koranyi), and more recently by W. Woess), but this opens a challenging new problem if the same techniques are used to deal with polyharmonic functions. The boundary representation theorems of eigenfunctions will be used as a tool to obtain the same for polyharmonic functions on any tree, thereby bringing to trees a result of Almansi as old as 1898, and more recent ones for the isotrpic case by Cohen, Colonna, Gowrysankharan and Singman. But the new approach is that this can be done avoiding case by case detailed calculations and using instead a clever differentiation method for generalized Poisson kernel. Then this idea will hold much more generally, and we'll be able to bring the result to the hyperbolic disk and even to higher rank symmetric spaces: a nice outcome for a theory originally based on discrete structures. Of course there is no interest in boundary representations unless they are applied, and we shall apply them to find the radial (and non-tangential) limit behavior of polyharmonic functions on trees and on the hyperbolic disk.

The spherical representations that we are going to study will establish a similarity between Aut H and SL(2,R). P. Cartier's original approach to the theory of representations of the 2x2 projective linear group over a p-adic field is based on the realization of the group as a group of isometries of a homogeneous tree of order p + 1. The interplay between $PSL(2, R), PSL(2, Q_p)$ and homogeneous trees leads to an additional innovative idea (to be developed jointly with Florin Radulescu), aimed to study the interplay between the representations of the Hecke algebras on various vector spaces (e.g. automorphic forms or Maass forms) and the unitary representation theory of the group $G = PSL(2, \mathbb{Z}[1/p]) \subseteq PSL(2, Q_p)$. Weak containment in the left regular representation of $PSL(2, Q_p)$ corresponds to the validity of the Ramanujan–Petersson conjectures for the Hecke operators on the vector spaces taken in consideration. In particular for automorphic forms this would give a first harmonic analysis proof of the Ramanjan estimates for automorphic forms (proven before by Deligne [16] by an indirect method). In the case of Maass forms this would allow to prove completely new estimates.
There are two interesting (exotic) topologies on the group algebra of G coming from the restrictions of the left regular representations of PSL(2,R), respectively $PSL(2, Q_p)$, to G. A method of passing from the restriction of a discrete series unitary representation of $PSL(2, R)$ to a unitary representation of $PSL(2, Q_p)$ is available. This transfer of the unitary representations amounts to the calculation of the structure of the corresponding representation of the Hecke operators, which are "block" matrix coefficients of the new representation of $PSL(2, Q_p)$. The matrix coefficients are spherical functions on $SL(2, Q_p)$ and their interpretation is related to the eigenvalues of the Hecke operators (related to the Ramanujan–Petersson estimates of eigenvalues). We plan to establish an abstract and powerful formulation

of this transfer of unitary representations. The tool is the fact that the matrix coefficients of a discrete unitary representation of PSL(2,R) give an algebra-valued representation of the eigenvalues of the Laplacian associated to the homogeneous tree of degree p+1. Weak contaiment in the left regular representation of $PSL(2, Q_p)$ corresponds to the validity of the Ramanujan–Petersson conjectures for the Hecke operators on the vector spaces taken in consideration. In particular for automorphic forms this would give a first harmonic analysis proof of the Ramanjan estimates for automorphic forms (proven before by Deligne [16] by an indirect method). In the case of Maass forms this would allow to prove completely new estimates. This will give a new understanding of trace formulas for Hecke operators acting on automorphic forms.

So far we have dealt with homogeneous or semi-homogeneous trees or polygonal graphs. But many previous papers of the P.I. and this team have dealt with the spectral properties of the Laplacian (meant as the normalized adjacency operator) on networks, and related properties in analysis and potential theory. Therefore we complete our project with a research line that deals with graphs X that are not homogeneous, but whose degree is bounded. We consider infinite graphs, but we approximate them by a fixed exhaustion X_n made of finite subsets. We suppose further that X is obtained by a density-zero perturbation of some “homogeneous” network. As simple example we have density zero-perturbations of trees (i.e. non-amenable graphs for which interior (volume) cardinalities are not negligible compared with the boundary (surface) cardinalities). Let A be the adjacency operator (that becomes a multiple of the Laplacian in the case of homogeneous graphs; it is bounded and self-adjoint), and by A_n the adjacency operators associated to the fixed exhaustion. We shall study of the spectral properties of A for eigenvalues near its norm. This research line (main investigator Francesco Fidaleo) has natural physical applications to the Bose–Einstein Condensation in statistical physics and statistical mechanics.
The behavior of the resolvent of A when its eigenvalue lambda is near its norm coincides with the behavior of the resolvent of the Hamiltonian H for eigenvalues near zero: this shows the link between this line of research and the study of the Laplacian on homogeneous or semi-homogeneous trees, but as already observed now the graphs are not homogeneous.

Among the relevant properties to be studied (that depend on the chosen exhaustion which is however fixed), we mention: - The geometrical dimension of the graph X - The Perron-Frobenius density of X, describing the L2-behaviour of the Perron-Frobenius weight of A obtained as the limit of those of the A_n, provided the limit is uniquely determined (this happens for all examples under consideration on the basis of a natural choice of the exhaustion) - The recurrent/transient behavior of the (adjacency operator of the) graph, given by the finiteness of the Green kernel at the eigenvalue lambda in the limit when lambda tends to the norm of A - The appearance of the hidden spectrum of H in the above mentioned limit for the integrated density of states (the cumulative function describing the density of eigenvalues of H_{X_n} in the limit $X_n \approx X$, provided it exists

These properties are not only intrinsically significant, but they all have natural applications to the study of the Bose–Einstein Condensation (BEC) of free bosons associated to the so-called Pure Hopping Model on the graph X. In fact, the appearance of the BEC of the free bosons is governed by the behaviour of the

Planck distribution. This research line produces a significant result in physics: that the transience/recurrence of the adjacency operator on the graph (that is a spectral property) is equivalent to the occurrence/non-occurrence of the Bose–Einstein condensation.

2. State of the art

This project deals with infinite networks, but an important part is devoted to trees and polygonal graphs having uniformly spanning trees. What is unique to this special environment is the existence of a large group of isometries: the action of this group gives rise to convolutions and harmonic analysis. So, the first research line is aimed to study harmonic analysis on trees and representation theory of their group of automorphisms via a geometric approach. More generally, we can give up the group of isometries and study in this wider context also the spectral properties of nearest neighbors operators on general trees and graphs of bounded valency. The research has to do with harmonic analysis, the Hecke algebra, the geometry of trees (and graphs) given by the Laplace operator on homogeneous and semi-homogeneous trees and the adjacency operator on graphs of bounded degree (i.e., valency). The main tools are the horospherical fiber bundle and the Poisson kernel therein, and the spectra of the discrete Laplacian and of the adjacency matrix.

Harmonic analysis and spectra of the Laplace operator on an infinite homogeneous tree T and representation theory of groups acting thereon have been studied in many articles and books. In all these references and in our preceding works, we have studied harmonic analysis on trees by looking at algebras of functions on the set V of vertices of a homogeneous tree and the action of subgroups of the automorphism group on V. One of the main goals of the present project is to give a completely different and innovative approach to harmonic analysis on trees, starting not from the set of vertices, but from horospheres on T and their fiber bundle $\mathcal{H}$, whose base is the boundary ot T (the set of tangency points of horospheres) and whose fibers, isomorphic to $\mathbb{Z}$, are given by the horospherical index. This mix of analysis and geometry provides results not only in analysis and operator theory but also in integral geometry (the horospherical fiber bundle and its automorphisms, range and inversion of the various Radon transforms). A significant advantage of this approach is that $\operatorname{Aut}\mathcal{H}$ is much larger than $\operatorname{Aut}T$, and in particular has a non-trivial center, that contains a group $A \approx \mathbb{Z}$ of parallel shifts along fibers. This group can be used, in harmonic analysis on trees, much in the same way as the Cartan subgroup of the Iwasawa decomposition for semi-simple groups and their symmetric spaces. This fact will allow us to realize the spherical representations of Aut T as induced representations from A, in the sense of Mackey.

A tree is a simplicial complex whose simplexes are vertices and edges (and also flags, i.e. oriented edges). Then homogeneous and semi-homogeneous trees are the lowest rank case of the simplicial complexes defined by the Bruhat–Tits buildings [5, 51] in their combinatorial definition, whose groups of automorphism at rank n is $PSL(n, \mathbb{Q}_p)$. Therefore this project is a preliminary step towards a revision in terms of integral geometry of the fundamental work of Macdonald on the spherical Fourier transform on semisimple p-adic groups.

Let us denote by V and E the set of vertices and of edges. We shall consider Radon transforms over horospheres $\mathcal{R}_V$, $\mathcal{R}_E$ related to these simplexes. If f is a function on V or E, its Radon transform $\mathcal{R}f$ is the sum of the values of f on each horosphere. We shall obtain applications to harmonic analysis, like the study of spherical functions on the edges of the tree and related new analytic families of unitary and uniformly bounded representations of $\operatorname{Aut} T$, in the same spirit of the representation theory for $SL(2, \mathbb{R})$, and inversion formulas for $\mathcal{R}_V$ and $\mathcal{R}_E$. The main tool that links spherical functions ot integral geometry is the Poisson transform, that is a boundary integral representation whose integral kernel, the Poisson kernel, can be lifted to a function on the horospherical fiber bundle, actually a character of A. The spherical representations will be realized as induced representations from the characters of this group of dilations of the fiber bundle.

The setup becomes more intriguing in the framework of semi-homogeneous non-homogeneous trees, where we shall still be able to describe spherical functions as Poisson integrals, but they will not be induced from characters of A, and indeed, integral geometry will not be adequate to produce them explicitly: we shall need probability theory, namely the recurrence relations of the first visit probabilities of the random walk generated by the Laplacian on the semi-homogeneous tree.

Long ago, P. Cartier's followed a combinatorial approach to the theory of representations of $PGL_2(\mathbb{Q}_p)$ (the two by two projective linear group over a $p-$adic field $\mathbb{Q}_p$) by realizing this group as a group of isometries of a homogeneous tree T of order $p+1$. In this context, we shall study the interplay between the representations of the Hecke algebras on various vector spaces (e.g. automorphic forms) and the unitary representation theory of the group $G = PSL(2, \mathbb{Z}[(1/p)),$ where p is a prime number. In preceding work we have exhibited a method of passing from the restriction of a discrete series unitary representation of $PSL(2, \mathbb{R})$ to a unitary representation of $PSL(2, \mathbb{Q}_p)$. This transfer of the unitary representations amounts to the calculation of the structure of the corresponding representation of the Hecke operators, that are related to spherical functions on $SL(2, \mathbb{Q}_p)$ and the eigenvalues of the Hecke operators. In this project we plan to establish an abstract formulation of this passage from unitary representation of $PSL(2, \mathbb{R})$ to $PSL(2, \mathbb{Q}_p)$. This is based on the fact that the matrix coefficients of a discrete unitary representation of $PSL(2, \mathbb{R})$ give an algebra valued representation of the eigenvalues of the Laplacian associated to the homogeneous tree of degree $p+1$.

For general networks, we do not have groups of isometries and harmonic analysis, but we still have adjacency operators and their spectral theory. In this environment, our project aims to describe a classical phenomenon in statistical mechanics (the Bose–Einstein condensation) in terms of the spectrum of the adjacency operator. The main tools and properties are the geometric dimension of the network, the Perron–Frobenius theorem, the recurrence or transience of the adjacency operator, the Green kernel at the eigenvalue 1 (that is, again, potential theory) and the hidden spectrum of the Hamiltonian operator.

3. The research lines of the project

3.1. Harmonic analysis on homogeneous trees via integral geometry. Harmonic analysis on a tree T and representation theory of groups acting on it have been studied in many articles and books: see, for instance, [7, 14, 17–20, 22] and their bibliographies. In all these references, harmonic analysis was studied by looking at algebras of functions on the set V of vertices of a homogeneous tree (or semi-homogeneous, in the last reference) and the action of subgroups of the automorphism group on V. The goal of the first research line of the present project (with main collaborators Simon Gindikin and Enrico Casadio–Tarabusi) is to give a different approach to harmonic analysis on trees, inspired by [25], starting not from the set of vertices, but from horospheres on T and their fiber bundle $\mathcal{H}$, whose base is the boundary ot T (the set of tangency points of horospheres) and whose fibers, isomorphic to $\mathbb{Z}$, are given by the horospherical index. Our motivation for this approach is that $\operatorname{Aut}\mathcal{H}$ is much larger than $\operatorname{Aut}T$, and in particular has a non-trivial center, that contains a group $A \approx \mathbb{Z}$ of parallel shifts along fibers that can be used, in harmonic analysis on trees, much in the same way as the Cartan subgroup of the Iwasawa decomposition for semi-simple groups and their symmetric spaces.

The horospherical transform, or Radon transform, introduced in [?Radon] for Euclidean spaces, was later extended to complex classical groups [24, 25] and symmetric spaces (see [29] for references). It has also been extended to trees, as the summation over the vertices in each horosphere: it was introduced on homogeneous trees in [7] and studied in [3];later on, it was extended to non-homogeneous trees in [9, 11].

Our motivation to study Radon transforms and harmonic analysis based on horospheres for functions defined not only on vertices of a tree, but also on edges and flags, is the following. A tree is a simplicial complex whose simplexes are vertices, edges and flags. Then homogeneous and semi-homogeneous trees are the lowest rank case of the simplicial complexes defined by the Bruhat–Tits buildings (in their combinatorial definition). Therefore this project is a preliminary step towards a revision in terms of integral geometry of the fundamental work [35] on the spherical Fourier transform on p-adic groups.
We shall consider Radon transforms over horospheres $\mathcal{R}_V$, $\mathcal{R}_E$, $\mathcal{R}_F$ related to these simplexes and derive applications to harmonic analysis, like the study of analytic families of unitary and unitary bounded representations of $\operatorname{Aut}T$, in the same spirit of the representation theory for $SL(2,\mathbb{R})$.

Here is an outline of our approach. We start with the fiber bundle $\mathcal{H}$. We could even start with an abstract bundle $\mathcal{H}$, not originating from a tree, since the tree can be reconstructed from $\mathcal{H}$ and a choice of nested families of subsets of its base $\mathcal{H}/A$, and the boundary Ω can be obtained as $\mathcal{H}/A$, but for simplicity we shall introduce $\mathcal{H}$ as the bundle of horospheres in a homogeneous or semi-homogeneous tree T). Then we consider some space of functions $\mathfrak{U}$ on $\mathcal{H}$ and decompose it as direct integral over $\widehat{A} \approx \mathbb{T}$ of subspaces $\mathfrak{U}_\sigma$ invariant under the action of the parallel shift group A, where A acts via its characters σ. Since A commutes with $\operatorname{Aut}T$, the spaces $\mathfrak{U}_\sigma$ are invariant under $\operatorname{Aut}T$, and the action of $\operatorname{Aut}T$ (or its subgroups) gives rise to representations π_σ of $\operatorname{Aut}T$ and its subgroups. These representations can be naturally realized on the boundary $\Omega \approx \mathcal{H}/A$, but for this goal it is more appropriate to give a different model for Ω, by realizing it as any of the *special sections* of $\mathcal{H}$. The special sections are in one to one correspondence with the vertices v: the special section Σ_v consists of all horospheres that pass through v (although this definition is obviously based upon starting with the set of vertices, these special sections can be also reconstructed if one starts with an abstract fiber bundle $\mathcal{H}$ and not with a specific tree). The choice of a special section induces a global chart on $\mathcal{H}$, that is, it gives rise to specific choice of integer coordinates on the fibers: the special section is at level 0. This allows to write explicitly the representations π_σ in these coordinates. These representations, called spherical representation of $\operatorname{Aut}\mathcal{H}$, extend the spherical representations of $\operatorname{Aut} T$ and of its discrete subgroups acting simply transitively on T, studied in [19, 20] and more generally in [22], that were proved in these references to be irreducible.

This setup gives a completely different and new flavor and insight to the theory of spherical representations and Poisson transforms develeoped in [20]. Observe that the special section Σ_v is endowed with a unique invariant normalized measure ν_{v_0} under the action of the isotropy subgroup $K_v \subset \operatorname{Aut} T$ of v. This allows us to introduce a natural intertwining operator $\mathfrak{R}_v$ from functions on $\mathcal{H}$ to functions on V, given by integration with respect to ν_v on one side, and radialization around v on the other side. When we choose a different special sectio Σ_{v_0}, the invariant probability measure ν_v turns out to be absolutely continuous with respect to ν_{v_0}. The Radon–Nikodym derivative is homogeneous along the fibers, and all homogeneous functions along the fibers are related to it: in the fiber coordinate n their values are $\sigma(n)$, where σ are the characters of A (let us write $\sigma(n) = \sigma_z(n) = e^{izn}$ for some complez number z). Therefore, for each character σ_z of A, we obtain a Poisson transform from functions on Σ_{v_0} to function on V, with integral kernel K^z, where $K(x, \omega)$ is the usual Poisson kernel. The image of the Poisson transform consists of eigenfunctions (with eigenvalues depending on σ) of the generator of the *Hecke algebra* (the algebra under the convolution defined by $\operatorname{Aut} T$) of functions radial around v_0). This operator is usually called the *Laplace operator* μ_{v_0}. It is known that this map is surjective from finitely additive measures on Σ_{v_0} (sometimes called *boundary distributions* in the literature) and eigenfunctions of μ_{v_0}. Some eigenvalues are positive-definite with respect to $\operatorname{Aut} T$, and for their eigenvalues the representations π_σ are unitary.

In the direct integral decomposition of the function $f \in \mathfrak{U}$ into the spaces $\mathfrak{U}_\sigma$, the component f_σ in $\mathfrak{U}_\sigma$ defines the value of the vertex-spherical Fourier transform $\mathcal{G}^V_{v_0} f(\sigma, \omega)$ at the character σ and the fiber ω. If the points of the horospherical fiber bundle $\mathcal{H}$ are realized as sets of vertices (horospheres in V), this allows to introduce a horospherical transform (usually called Radon transform in the literature) from functions on V to functions on $\mathcal{H}$, given by summation over each horosphere. The interplay between the spherical transform, the horospherical transform and the Fourier series expansion in $\widehat{A} \approx \mathbb{T}$ leads to the Fourier slice theorem, that provides a Plancherel formula for the spherical transform over $\mathbb{T}$. The integral weight that appears in the Plancherel formula is related to a function that has an important analogue in harmonic analysis on semisimple groups, the Harish-Chandra $c-$function, that was originally computed on trees in [17, 20].

Following [3] and introducing a suitable space of test functions $\mathcal{S}(V)$ on V (*Schwartz class*, small enough that its dual $\mathcal{S}'$ (space of distributions on V) contains all eigenfunctions of the Laplace operators corresponding to unitary representations, we shall extend the horospherical transform to distributions and show that the $c-$function is an eigenvalue. By the same token we also prove a Paley–Wiener theorem for the spherical Fourier transform. We define a Schwartz class $\mathcal{S}(\mathcal{H})$ also on the horospherical fiber bundle, and show that the Radon transform maps $\mathcal{S}(V)$ to $\mathcal{H}$. Then we consider the geometric back-projection $\mathcal{R}^*$ from function on $\mathcal{H}$ to functions on V and show that $\mathcal{R}^*\mathcal{R}$ is a convolution operator, we compute its symbol and use it to provide an inversion formula for $\mathcal{R}$.

Spherical representations of a free group or free product (regarded as a subgroup of $\mathrm{Aut}\, T$) on functions on vertices were studied on the basis of an analogy between the homogeneous tree of degree $p+1$ and the representation theory of $SL(2,\mathbb{R})$ as follows. P. Cartier's combinatorial approach to the theory of representations of $PGL_2(\mathbb{Q}_p)$ (the two by two projective linear group over a $p-$adic field $\mathbb{Q}_p$) is based on the realization of the group as a group of isometries of a homogeneous tree T of order $p+1$ [7]: indeed, this tree is obtained by an explicit combinatorial constructions that defines as vertices the lattices of $\mathbb{Q}_p^2$ mod its ring of integers $\mathbb{Z}_p^2$ (see also [52]). In the case of edges, on the boundary Ω of T we can consider the hitting probability measure ν by means of the simple random walk. The action of $PGL(2,\mathbb{Q}_p)$ on T induces an action on Ω, and the measure ν is quasi-invariant with respect to the action of $PGL(2,\mathbb{Q}_p)$. That is, letting $\nu_g(E)=\nu(gE)$, the measure ν_g is absolutely continuous with respect to ν, and one may define a Poisson kernel as the Radon-Nikodym derivative $K(g,\omega)=d\nu_g/d\nu(\omega)$, for $\omega\in\Omega$. Finally, if $z\in\mathbb{C}$, one defines the spherical representations of $PGL_2(Q_p)$, on the space of simple functions on Ω, by the formula

$$(\pi_g^V(\xi))(\omega)=K^z(g,\omega)\xi(g^{-1}\omega).$$

Spherical functions are then obtained integrating $K^z(gi,\omega)$ over the boundary. These representations are called *spherical representations*, introduced, in the case of spaces of functions on vertices, in [19] and studied in [20, 22] and many other papers. It was shown in [19, 19] that, if $0\leqslant \mathrm{Re}\, z\leqslant 1$, they are an analytic family of uniformly bounded representations of the free group, unitary if $\mathrm{Re}\, z=\frac{1}{2}$ or unitarizible if $\mathrm{Im}\, z=k\pi/\ln q$ (principal and complementary series), and that the principal and complementary series are irreducible. A discrete series of $\mathrm{Aut}\, T$ was studied by Olchanskyi (see also [18]).

All of this will also be done for edges. In particular, this will yield a Plancherel formula also for harmonic analysis on edges, hence a decomposition of the regular representation of $\mathrm{Aut}\, T$ also in terms of edge-spherical functions. We have already seen that the construction of spherical representations π_z starting with the horospherical fiber bundles requires not only a character $\sigma_z(n)=q^{nz}$ of the Cartan subgroup A and a group of automorphisms acting on the bundles, but also the choice of a special section in the fiber bundle, that is, a choice of reference vertex v_0 or reference edge e_0. We shall prove that the vertex-horospherical bundle $\mathcal{H}_V$ and the edge-horospherical bundle $\mathcal{H}_E$ are isomorphic. However, we shall also prove that there exists no automorphism from one to the other that maps special sections to special sections. Therefore, in principle, the vertex-spherical representations and the edge-spherical representations induced by the same character of A might be inequivalent. We plan to show that these representations actually do not depend on the choice of reference vertex or edge, and therefore from the choice of special section. But in order to prove that they are equivalent, a sufficient condition would be to reformulate these representations not as based on a special section, but on a general section: that is, to reformulate them in terms of a global chart but not on an individual reference point, so that the character $\sigma_z(n)$ would have to be parameterized as $\sigma_z(n_\omega)$, depending on the fiber. This means to exprss the representations purely on the basis of elements of the horospherical fiber bundle, not on the choice of a vertex or edge. If this challenging step can be completed, then π_z^E would be equivalent to π_z^V. In particular, the edge-spherical representations of the principal and complementary serieswould be topologically irreducible, because so are the

vertex-spherical representation when restricted to a simply transitive subgroup of $\operatorname{Aut} T$, as proved in [19, 20].
Moreover, the spectrum of the Laplace operator on $\ell^p(E)$ can be completely described in the same way as it was done on $\ell^p(V)$ in the references mentioned above, except for a shift due to the fact that the recurrence relation for edge-spherical functions will still be a second order difference equation but with three terms instead of two.

3.2. **Representations of p-adic semisimple groups arising from the geometry of trees and the Ramanujan–Petersson conjectures.** The interplay between $PSL(2,\mathbb{R})$, $PSL(2,\mathbb{Q}_p)$ and homogeneous trees leads to the second research line of the project (main investigator Florin Radulescu), aimed to study the interplay between the representations of the Hecke algebras on various vector spaces (e.g. automorphic forms or Maass forms) and the unitary representation theory of the group $G = PSL(2,\mathbb{Z}[1/p]) \subseteq PSL(2,Q_p)$, where p is a prime number.
There are two interesting C^*-algebra (exotic) topologies on the group algebra of G coming from the restrictions of the left regular representations of $PSL(2,\mathbb{R})$ respectively $PSL(2,\mathbb{Q}_p)$ to G. In preceding work we have exhibited a method of passing from the restriction of a discrete series unitary representation of $PSL(2,\mathbb{R})$ to a unitary representation of $PSL(2,\mathbb{Q}_p)$. This transfer of the unitary representations amounts to the calculation of the structure of the corresponding representation of the Hecke operators, which are a "block" matrix coefficient of the new representation of $PSL(2,\mathbb{Q}_p)$. The matrix coefficients are spherical functions on $SL(2,\mathbb{Q}_p)$ and their interpretation is related to the eigenvalues of the Hecke operators (related to the Ramanujan–Petersson estimates of eigenvalues). In this project we plan to establish an abstract formulation of this passage from unitary representation of $PSL(2,\mathbb{R})$ to $PSL(2,\mathbb{Q}_p)$. This is based on the fact that the matrix coefficients of a discrete unitary representation of $PSL(2,\mathbb{R})$ give an algebra valued representation of the eigenvalues of the Laplacian associated to the homogeneous tree of degree $p+1$.

More precisely, in previous research [56] we analized the unitary representations of $PSL(2,\mathbb{Q}_p)$ that are obtained, through the process described above, starting with unitary representations in the (analytic) discrete series of $PSL(2,\mathbb{R})$. We were able to identify the character [8, formula 13] of the unitary representation obtained this way. This computation was independently confirmed by number-theoretic methods in [59]. This gives a better understanding of trace formulas for Hecke operators acting on automorphic forms.

In this project we expect to further extend this interplay between operator algebras representations of the Hecke operators and the corresponding number-theoretic statements. Weak containment in the left regular representation of $PSL(2,Q_p)$ corresponds to the validity of the Ramanujan–Petersson conjectures for the Hecke operators on the vector spaces taken in consideration. In particular for automorphic forms this would give a first harmonic analysis proof of the Ramanjan estimates for automorphic forms (proven before by Deligne [16] by an indirect method). In the case of Maass forms this would allow to prove completely new estimates.
Analyizing the characters of the transferred representation allows (as in the proof of the Plancherel formula for p-adic groups) to determine the expression of the character of the transferred representations as a limit of convex combinations of characters of irreducible representations and hence this gives a method to decide if

the transferred representation is tempered (which corresponds to the validity of the Ramanujan–Petersson estimates for the initial vector space). In certain cases the calculations of the characters were independently confirmed by number theoretic methods, involving traces of Hecke operators in [59] (based on methods previously considered by D. Zagier).

A second research goal is the following. Let B be the boundary of the tree associated to $PSL(2, \mathbb{Z})$ and let Ω the boundary associated to the homogenuous tree of degree $p+1$, p a prime number. Jointly with F. Radulescu e J. Bassi, we'll analyze the reduced crossed product algebra $C^*(B \times \Omega) \rtimes PSL(2, \mathbb{Z}[1/p])$ and its relation to the Hamana boundary ([Ha]). The goal is to find a concrete realization of the nuclear envelope of the exact reduced group algebra of $PSL(2, \mathbb{Z}[1/p])$, and its action on its Furstenberg boundary. The tools will be the techniques developed in [4, 37], Also, by a well known result, probably first noticed by Ihara, $PSL(2, \mathbb{Z}[1/p])$ is a lattice in $PSL(2, R) \times PSL(2, Q_p)$ and hence we can use the properties of the boundaries of $PSL(2, R)$ and $PSL(2, Q_p)$.

3.3. **Semi-homogeneous trees, Poisson kernels, potential theory and Poisson representation of λ-polyharmonic functions.** Most of the results can be also considered in the setting of semi-homogeneous trees. In this setting, however, the Poisson kernel $K(x, \omega)$ will still be a functions on horospheres (that is, constant on each horosphere tangent at ω), but the kernel for the Poisson transform at the eigenvalues different from 1 may not necessarily be of the type K^z. Then this kernel may be hard to compute with techniques of integral geometry or harmonic analysis, but can be computed with probabilistic techniques (the study of the random walk induced by the Laplacian given by the adjacency operator on the semi-homogeneous tree). This fact is likely to yield a new spectral theory for this Laplacian, and different spherical functions and representations, perhaps reducible.

Here we have another intriguing set-up, where networks different from trees start appearing. For simplicity, in this discussion we focus attention on vertex-spherical representations. The group $\operatorname{Aut} T$ is not transitive on the vertices of a semi-homogeneous tree, since there are two alternating homogeneity degrees that must be preserved by automorphisms. So a natural notion of convolution, defined by the choice of a simply transitive group of isometries, is not available, and we only have convolution in $G = \operatorname{Aut} T$ expressible as convolution of functions on the tree regarded as G/K where K is the stability subgroup of a vertex, hence of right-K-invariant functions, but only if one of the two functions is bi-K-invariant, i.e. radial. This would indeed be enough to study the spectrum of the isotropic Laplacian, that is radial, but not the spherical representations, that require the action of a group of isometries from a reference vertex to any other vertex.

But then, let us restrict attention for the moment to functions in V: the Laplacian is the operator μ_1 of average on neighbors, but the significant operator is the two-step average μ_2, that is essentially μ_1^2 (up to a linear combination with the identity operator μ_0 and is based on jumps of length 2. Now, μ_2 has non-zero jumps of length 2, but the set S_2 of vertices of distance 2 from a reference vertex v_0, regarded as vertices in a new graph, has loops: there are pairs (v_1, v_2) in S_2 such that $d(v_1, v_2) = 2$. Therefore the Cayley graph of the group generated by μ_2 is not a tree. This graph, for a semi-homogeneous tree of degrees q_0, q_1, is a polygonal graph having a spanning tree made of complete polygons of cardinalities q_0 and q_1 in an alternating way. Harmonic analysis on these graphs was studied in [31]. A

simply transitive subgroup on this Cayley graph is $\Gamma = \mathbb{Z}_{q_0} * \mathbb{Z}_{q_1}$. The convolution defined by this group gives rise to spherical representations of Γ. The spectral theory of μ_2 is similar to the spectral theory of μ_1 on a homogeneous tree, because the actions of these two operators have similar recurrence relations. However, the spectral theory of μ_1 on T_{q_0,q_1} can be quite different: we plan to unveil it in full. We expect that the spectrum on ℓ^1 (or equivalently ℓ^∞) will still be an ellipse as in the homogeneous environment, but that the shape of spectrum on the other ℓ^p spaces will be considerably different. Here neither convexity of the spectrum nor connectedness is granted: it will be interesting to find a critical index p at which the spectrum becomes disconnected. Even more intriguing may be the study of the spectra of the Laplacian on functions on the set E of edges of semi-homogeneous trees, because on edges the group $\operatorname{Aut} T$ is transitive on semi-homogeneous non-homogeneous trees. Hence in this environment we cannot reduce attention to a step-2 Laplacian whose spectral theory is the same as in the usual homogeneous setup: yet the set E, regarded as a Cayley graph, is a polygonal graph but not a tree, because all edges joining at a vertex touch one another, hence the Cayley graph must contain complete polygons (note that this is already so for $E(T)$ of a homogeneous tree T).
This part of the project will be done by the principal investigator jointly with E. Casadio Tarabusi.

This, in turn, leads us to consider the interplay between spectral theory of the Laplacian and the Poisson kernel. This was done for general trees in [7], but only for positive eigenvalues. Then it was done for the isotropic nearest neighbor Laplacian invariant under $\operatorname{Aut} T$ on a homogeneous tree in [36], and for non-isotropic but invariant Laplacians in [22]. The book [53] gives an excellent account of the general case of Laplacians on not homogeneous trees, and in the recent paper [49] the Poisson representation of eigenfunctions of the Laplacian was extended to trees not even locally finite and to all eigenfunctions. At the same time, the last reference proves a similar Poisson representation for the space of polyharmonic functions, that is the kernel of $(\Delta - \lambda\mathbb{I})^n$ for an arbitrary integer n. In order to pass from the Poisson representation theorem for harmonic functions and λ-eigenfunctions of Δ to the representation of λ-polyharmonic functions, what was used is the existing Poisson representation of λ-eigenfunctions and a clever differentiation with respect to λ. These methods can be used also in a smooth, non-discrete environment. We plan to obtain the same Poisson representation theorem for polyharmonic functions of the Laplace–Beltrami operator in a (rank one) hyperbolic space, and possibly also for the algebra of invariant differential operators on higher rank symmetric spaces, since in these domains a Poisson representation of λ-eigenfunctions of invariant differential operators is available [29, 32]. This results is particularly attractive because it completes a classical problems: the Poisson representation of polyharmonic functions (only for the eigenvalue 0 and on $\mathbb{R}^k$) have been proved very long ago, by Almansi, in 1898 [1].
Once a Poisson boundary representation is achieved, its natural use is to prove a boundary limit theory, let us say a radial limit theory for functions with continuous boundary data. This is well known in the case of harmonic functions on the hyperbolic disc. Since the Laplace–Beltrami operator in the hyperbolic disc coincides with the Euclidean Laplacian, this goes back to the classical Fatou' convergence

theorem for harmonic continuation inside the disc of continuous boundary functions. However, for eigenvalues λ different from 0, the hyperbolic λ-Poisson kernel is a complex power of the usual Poisson kernel, but the Euclidean counterpart is not known. For λ-polyharmonic functions of order $n > 1$, no limit theorem is available directly, even for the eigenvalue 0, because the Poisson representation has an additional polynomial factor that diverges at the boundary. In the case of a homogeneous tree and the isotropic Laplacian, the right limit theorem was proved many years ago in [15], by means of a suitable normalization. More recently, W. Woess handled this problems for homogeneous trees and the invariant isotropic Laplacian, by showing that the correct normalization is given by dividing by the n-th derivative with respect to λ of the spherical function ϕ_λ corresponding to the eigenvalue λ. We plan to prove this result for λ-polyharmonic functions of the Laplace–Beltrami operator on the hyperbolic disc, and possibly for invariant differential operators in higher rank symmetric spaces. The normalization factor should be the derivative with respect to λ of the spherical function ϕ_λ, as on a homogeneous tree, but in the smooth case it is much harder to compute: the derivative is not known and even its asymptotic behavior is unknown, and depends on complicated estimates on the rate of growth of the hypergeometric function.

The part of our project presented in this Subsection proposes (and hopefully solves) several problems of potential theory on smooth classical domain (hyperbolic spaces and higher rank symmetric spaces) by the inspiration coming from trees (and occasionally graphs): this way of proceding is innovative, because normally results on trees are proved by analogy with already known results on the disc, and not the other way around.

SubS:Bose-Einstein

3.4. **Spectral properties of graphs, statistical mechanics and the Bose–Einstein condensation.** So far we have dealt with homogeneous or semi-homogeneous trees. But many previous papers of members of this team have dealt with the spectral properties of the *Laplacian* (meant as the normalized adjacency operator) on non-homogeneous trees or graphs and related properties in analysis and potential theory: see the papers of M.Picardello [17, 30, 31, 38–50] and, for the adjacency operator (un-normalized Laplacian) those by F. Fidaleo [61–65]. Therefore we complete our project with a research line that deals with graphs X that are not homogeneous, but whose degree is bounded. We consider infinite graphs, but we approximate them by a fixed exhaustion $\{X_n\}_{n\in\mathbb{N}} \subset 2^X$ made of finite subsets. Suppose further that X is obtained by a density zero perturbation of some "homogeneous" network. As a simple example we have the so-called comb graphs (cf. [64, ?Fifaleo&Guido&Isola]), but also density zero perturbations of the Cayley trees (i.e. non-amenable graphs for which interior (volume) effects are not negligible compared with the boundary (surface) effects (cf. [61, 63])).

Let V as usual be the set of vertices of X and A be the adjacency operator $A \in \mathcal{B}(\ell^2(V))$ (that becomes the Laplacian in the case of homogeneous graphs; it is bounded and self-adjoint). Denote by $\{A_n\}_{n\in\mathbb{N}}$ the adjacency operators associated to the fixed exhaustion. We shall study of the spectral properties of the adjacency operator A for eigenvalues near its norm. This research line (main investigator Francesco Fidaleo) has natural physical applications to the Bose–Einstein Condensation.

Consider the operator $\Delta = \mathbb{I} - \frac{A}{\|A\|}$, closely related to the usual Laplacian (indeed, this is the definition of Laplacian on a graph as given in [53] and numerous other references below). Then the behavior of the resolvent of A when its eigenvalue λ is near $\|A\|$ coincide with the behavior of the resolvent of $H := \|A\|\Delta$ for eigenvalues near zero: this shows the link between this line of research and the study of the Laplacian on homogeneous or semi-homogeneous trees, but as already observed now the graphs are not homogeneous.

Among the relevant properties to be studied (that depend on the chosen exhaustion which is however fixed), we mention:

(i) The geometrical dimension $d_G(X)$ of the graph X (cf. [64]).
(ii) The Perron-Frobenius density $d_{PF}(X)$ of X, describing the ℓ^2-behaviour of the Perron-Frobenius weight of A obtained by the limit of those of the A_n, provided the limit is uniquely determined (this happens for all examples under consideration after a natural choice of the exhaustion, see [63, 64]).
(iii) The recurrent/transient behavior of the (adjacency operator of the) graph, given by the finiteness of the Green kernel at the eigenvalue λ in the limit $\lambda \downarrow \|A\|$ [53] (see also [61, 63–65]
(iv) The appearance of the *hidden spectrum of* H for $\lambda < \|A\|$, $\lambda \approx \|A\|$) for the *integrated density of states* (the cumulative function describing the density of eigenvalues of H_{X_n} in the limit $X_n \uparrow X$, provided it exists (see [62]).

These properties are not only intrinsically significant, but they all have natural applications to the study of the Bose–Einstein Condensation (BEC) (BEC for short) of free bosons made of the so-called Bardeen-Cooper pairs, associated to the so-called *Pure Hopping Model* on the graph X whose dynamics, and therefore the statistical properties, is described by the Pure-Hopping Hamiltonian

$$H_{PH} := -J_o \sum_{x,y \in VX} A_{x,y} a^\dagger_x a_y \,.$$

(here, $a^\dagger_x$ denotes the boson creator corresponding to the place x on the graph and $J_o > 0$ is a coupling constant describing the mobility of the bosons on the lattices).

In fact, the appearance of the BEC of the free bosons is governed by the behaviour of the Planck distribution

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \,.$$

It naturally involves the operator $\left(e^{\beta(H-\mu\mathbb{1})} - \mathbb{1}\right)^{-1}$, where $H = \|A\|\mathbb{1} - A$ is the one particle pure-hopping hamiltonian. For small energies and after using Taylor expansion, one heuristically gets

$$\frac{1}{e^{\beta(H-\mu\mathbb{1})} - \mathbb{1}} \approx \frac{1}{\beta(H-\mu\mathbb{1})} = \frac{1}{\beta((\|A\|-\mu)\mathbb{1} - A)} \equiv \frac{1}{\beta} R_A(\|A\| - \mu) \,.$$

Here, $\mu < 0$ is the chemical potential, and we can recognise that everything is therefore governed by the behaviour of the resolvent of A for $\|A\| - \mu =: l \approx \|A\|$.

In previous work we have studied the spectral properties of the adjacency operator and proved the deep connection of these properties with te appearance of the BEC for a wide class of networks. It is significant that the ideas that we have previously developed in this area of geometric spectral theory on networks have already been applied to physical experiments concerning the Bose–Einstein condensation, in particular in the nearby Department of Physics of the University of Rome Tor Vergata condicted by M. Cirillo et al. [68, 69]. In our project we shall also be involved in some of the related numerical simulations.

In this project we aim to provide the most general conditions on a general graph to assure the appearance of such very surprising spectral properties, and therefore the BEC.

Due to the non homogeneity, particles condensate even in configuration space. Therefore, by the presence of a huge amount of particles in very particular regions (i.e. those supporting the perturbation) of the space, the Coulomb interaction cannot be neglected. We than can argue that the a more realistic model cannot be described by the Pure-Hopping Hamiltonian (3.4) but by the so-called (non-quadratic) Bose-Hubbard Hamiltonian

$$H_{BH} = -J \sum_{x,y \in VX} A_{x,y} a_x^\dagger a_y + \frac{V}{2} \sum_{x \in VX} a_x^\dagger a_x (a_x^\dagger a_x - 1) - \mu \sum_{x \in VX} a_x^\dagger a_x .$$

Here, $J > 0$ describes the mobility of particles as before, $V > 0$ the Coulomb repulsion, and finally the chemical potential $\mu \leq 0$ fixes the density of the Bardeen-Cooper pairs of the model. Notice that, the Pure-Hopping model is recovered by putting $V = 0$.

Our challenging program is to connect the spectral properties of (the adjacency operator of) the network with the appearance of the BEC, even for the more realistic Bose-Hubbard model. This might provide a fundamental step concerning the long-standing problem in arguing that the appearance of the BEC is connected ONLY with the statistics and not with the strength/kind (repulsive or attractive) of the interaction occurring between the particles of the model.

We have already mentioned the fact that this research line on Bose–Einstein condensation based on spectral theory involves the adjacency operator, that is a constant multiple of the Laplacian only on graphs with constant degree. However, the arguments have a geometric nature, based on the geometry of the graph. This leads to an additional development: the computation of the perturbation of the spectrum of the Laplace operator on a homogeneous graph under small perturbations of the geometry of the graph. This subject is of interest for the geometry of graphs and for physical applications. It leads to the following new problems in our project:

(1) Although in this setup the hidden spectrum cannot appear, how does the perturbation change the integrated density of states in the limit of infinite volume (that is, the limit under the exhaustion)?
(2) What is, in the limit, the effect of the perturbation on the ℓ^2 behavior of the wave function (that is, the weight of the fundamental state)? What is the effect on the problem of transience vs. recurrence?
(3) How do these perturbational effects change if, instead than studying the spectrum of the adjacency operator, we consider the spectrum of the discrete Laplacian on the graph?

4. Tools for the expected results

We make here a summary of the various results expected, step by step, and the tools that will be used to prove them. Several of the contents of this part were stetted or hinted above: we make a list for the sake of clarity.

(1) Explicit computation of the spherical Fourier transform and the spherical functions for edges of a homogeneous tree T_{q+1} of valency $q + 1$. This

preliminary step, known for vertices, is not difficult but necessary. Tool: the recurrence relation for the powers of the Laplace operator on edges.

(2) Realization of the spherical representations on vertices and on edges of a homogeneous tree as induced representations from the dilation subgroup A of $\operatorname{Aut} H$ (Cartan subgroup): this makes use of the fact that the Poisson kernel is a nontrivial character of A, hence a generator for its dual group. Idea to prove the irreducibility of the spherical representations for the edges (for the vertices is already known):
since the edge-horospherical and edge-horospherical fiber bundles are isomorphic and the vertex-spherical representations are irreducible, the irreducibility of the edge-spherical representations might be proved in a nice elegant way (instead than by brute force) if we could show that it does not depend on the choice of section in the bundle used to define it, that is, on the global chart in the bundle. A preliminary step is to show that it does not depend on the choice of a special section. A direct computation could also be attempted, using methods of [20] or [22] transported to the simply transitive subgroup of $\operatorname{Aut} T$ acting on edges, that is $Z_{q+1} * Z_{q+1}$, but it would be considerably lengthy and painful, because the recurrence relation for edges has an extra term that complicates the combinatorics of words.

(3) Computing the spectrum of the Laplace operators for vertices and for edges via spherical functions (tat are their radial eigenfunctions). We shall show the the ℓ^p-spectrum is related to the belonging to ℓ^p of spherical functions (for $p \geqslant 2$), and then we'll determine the ℓ^p space to which a spherical function belongs by computing its asymptotic behaviour.

(4) Interplay between the Radon transforms and the spherical Fourier transform: the Fourier slice theorems. This step presents no difficulty, but is necessary for completeness of the results.

(5) Inversion formulas for the Radon transforms $\mathcal{R}$ on vertices and on edges in three ways:
    - directly;
    - via the Fourier slice theorem;
    - computing the back-projection $\mathcal{R}^*$, proving that $\mathcal{R}^*\mathcal{R}$ is a convolution operator, computing its spherical Fourier transform, i.e. its symbol, showing that it only vanishes at the boundary of the spectrum with a controlled rate of decay, and finally showing that its reciprocal gives rise to a bounded operator, hence to the inverse of $\mathcal{R}$.

    Tools: for the direct computation, we shall first obtain the inversion at the reference edge and then translate. Now, inversion at the referent edge is equivalent to inversion for radial functions. But for a radial function the problem requires only one parameter (the distance dorm the reference vertex, regarded as the horospherical index) hence we can provide solutions of a linear system by choosing sequence of coefficients iteratively. This leads to a continuum of inversion formulas that must all coincide on the image of the Radon transform (thereby providing uniqueness). Therefore the problem becomes the characterisation of the image of the edge-Radon transform. This will be done via Cavalieri conditions, similarly to the characterisation given for the image of the vertex-Radon transform in [9]. This method is a bit clumsy because it proceeds with linear systems and provides a set of

inversion formulas that become unique only upon restriction to the image of $\mathcal{R}$,. But it is very interesting because it is specific to trees, and could not even be formulated in a continuous setup.
Using the Fourier slice theorem is relatively obvious, but the computation for the inversion can be tricky.
Using the back-projection is more elegant. It will be easy to prove that $\mathcal{R}^*\mathcal{R}$ is a convolution operator on edges with respect to the convolution induced by the simply transitive subgroup $Z_{q+1} * Z_{q+1}$, and we foresee no difficulties in computing this convolutor explicitly. Therefore the task is to use the edge-spherical Fourier transform to invert this convolutor. We shall introduce a Schwarz class and a distribution space on the edges $E(T)$ and show that $\mathcal{R}^*\mathcal{R}$ maps the Schwartz class to distributions. The spherical Fourier transform gives the symbol of $\mathcal{R}^*\mathcal{R}$: we shall need to compute it explicitly, by making use of the spectral theory of the edge-Laplacian (that has to be described in detail: a tool for this is to compute the asymptotic of spherical functions and to prove Haagerup convolution estimate for functions on $E(T)$. Then we shall obtain the symbol of the inverse operator by showing that the order of zero of the symbol of $\mathcal{R}^*\mathcal{R}$ at the extreme points of the ℓ^2-spectrum is sufficient to provide invertibility in the distribution sense.

(6) The Poisson kernel at the eigenvalue λ of the Laplacians of a semi-homogeneous tree: proof of the fact that it is not a character of the Cartan subgroup A and cannot be computed via the horospherical fiber bundle. This will be done by making use of the recurrence relations given by the generating functions of the first visit probability of the random walk induced by the Laplacian on $E(T)$ regarded as a nearest neighbor isotropic transition operator.
(7) Computation of the Poisson kernel on semi-homogeneous trees via the recurrence relation of the Markov chains generated by the Laplacians. Same tools as in the previous point.
(8) Computation of the spherical functions on semi-homogeneous trees and the spectra of the Laplacians on ℓ^p. Tool: once the Poisson kernel has been computed in the previous point, we use it to integrate constants on the boundary. This yields the spherical functions. The computation will be lengthy but we do not expect problems.
(9) Analysis of the reducibility of the semi-homogeneous spherical representations. Tool: we shall try to adapt the methods of point 2 above.
(10) Statement and proof of the Poisson representation theorem for λ-polyharmonic functions of the Laplace–Beltrami operator on hyperbolic spaces, and of all invariant differential operators on higher rank symmetric spaces. Tool: we cannot use direct computation (like those for homogeneous trees and isotropic operators in [15]. We shall make use of a brilliant idea that can work on trees as well as hyperbolic spaces: a typical *lambda*-polyharmonic function of order m will be shown to be representable as a linear combination of the first m derivatives with respect to λ of the Poisson kernel at the eigenvalue λ.

(11) Statement and proof of boundary convergence theorems for n-polyharmonic functions for each eigenvalue of the Laplace–Beltrami operator on the hyperbolic space and of invariant differential operators on higher rank symmetric spaces, by choosing an appropriate normalization factor (it will consist of the n-th derivative of the spherical functions with respect to their eigenvalue, but their asymptotic behavior has to be computed). We shall bypass this computation by using the same clever differentiation trick as in the previous point. As a by-product, this will also yield an interesting corollary: the asymptotic behavior (when $r \mapsto 1$) of $\frac{F'}{F}$, where $F = F(\lambda, \lambda, 1, r^2)$ is Gauss' hypergeometric function and the derivative is with respect to λ.

(12) Transfer from unitary representations of $PSL(2, R)$ to the group $PSL(2, Q_p)$ (isomorphic to the automorphism group of the homogeneous tree with degree $p + 1$).
We shall consider the character of the transferred representation and to find its decomposition over the characters of the irreducible representations of $PSL(2, Q_p)$. The table of these characters is well known (see e.g [26, 60]): we shall proceed as in the determination of the decomposition involved in the Plancherel formula for $PSL(2, Q_p)$. Temperedness of the transferred unitary representation corresponds to the validity of the Ramanujan–Petersson estimates for the Hecke operators.

(13) Construction of separable, nuclear C^*-envelopes for gropus generalizing $PSL(2, Z[1/p])$ and their C^*-simplicity.
As in the theory of sofic groups representations we shall use methods from non-standard analysis (Loeb spaces) to determine the dynamics of the action of the groups. It can be proved that in the space of states one obtains by this construction the Maharam extension corresponding to the quasi-invariant measure on the boundary, and using this tool we expect to obtain the corresponding estimates, which then give information about the temperedness of the representation or estimates for the eigenvalues for the Hecke operators. We shall also use another recent result by L.Paunescu and F.Radulescu [54] , who have obtained as a corollary, a method to describe the action of Hecke operators in terms of countable, measurable equivalence relation,

(14) Spectral properties of the adjacency operator on bounded networks for eigenvalues near its norm.
It was show that, for the one-particle Hamiltonian consisting by (the opposite of) the discrete Laplacian of of a network obtained by additive, negligible perturbation of a standard one (i.e. periodic, as well as a Cayley tree), the hidden spectrum cannot appear, even if it this does not exclude the appearance of the BEC. However, we shall investigate the spectral properties of such a perturbed Laplacian. We shall study the Bose-Hubbard Hamiltonian by means of techniques recently developed by D. Buchholz [66, 67], with the aim of systematically handling non-quadratic hamiltonians by replacing the Weil algebra with the “Resolvent Algebra” in the context of the Canonical Commutation Relations.

Bibliography

Almansi [1] E. Almansi, *Sull'integrazione dell'equazione differenziale* $\Delta^{2n} = 0$, Annali di Matematica **2** (1899), 1–59.

ein&Casadio_Tarabusi&Cohen&Picardello [2] C. A. Berenstein, E. Casadio Tarabusi, J. M. Cohen, and M. A. Picardello, *Integral geometry on trees*, Amer. J. Math. **113** (1991), no. 3, 441–470, DOI 10.2307/2374835.

Betori&Faraut&Pagliacci [3] W. Betori, J. Faraut, and M. Pagliacci, *An inversion formula for the Radon transform on trees*, Math. Z. **201** (1989), no. 3, 327–337, DOI 10.1007/BF01214899.

BKKO [4] E. Breuillard, M. Kalantar, M. Kennedy, and N. Ozawa, *$C*$-simplicity and the unique trace property for discrete groups*, Publications Mathématiques IHES, vol. 126, 2017, pp. 35–71.

Brown [5] K. Brown, *Buildings*, Springer-Verlag, New York Berlin Heidelberg London Paris Tokyo, 1989.

Cartier-Symposia [6] P. Cartier, *Fonctions harmoniques sur un arbre*, Symposia Math., vol. 9, Ist. Naz. Alta Matem., Rome, 1972, pp. 203–270.

Cartier [7] P. Cartier, *Harmonic analysis on trees*, in Harmonic analysis on homogeneous spaces [Williams Coll., Williamstown, Mass., 1972], Proc. Sympos. Pure Math., vol. XXVI, Amer. Math. Soc., Providence, R.I., 1973, pp. 419–424.

Cartier-PSPM79 [8] ———, *Representations of p-adic groups: a survey*, in Automorphic forms, representations and L-functions, Part 1 [Oregon State Univ., Corvallis, OR, 1977], Proc. Sympos. Pure Math., vol. XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 111–155.

Casadio&Cohen&Colonna [9] E. Casadio Tarabusi, J. M. Cohen, and F. Colonna, *Range of the horocyclic Radon transform on trees*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 1, 211–234 (English, with English and French summaries).

CCKP [10] E. Casadio Tarabusi, J. M. Cohen, A. Korányi, and M. A. Picardello, *Converse mean value theorems on trees and symmetric spaces*, J. Lie Theory **8** (1998), no. 2, 229–254.

Casadio_Tarabusi&Cohen&Picardello [11] E. Casadio Tarabusi, J. M. Cohen, and M. A. Picardello, *The horocyclic Radon transform on nonhomogeneous trees*, Israel J. Math. **78** (1992), no. 2-3, 363–380, DOI 10.1007/BF02808063.

Casadio_Tarabusi&Gindikin&Picardello [12] E. Casadio Tarabusi, S. G. Gindikin, and M. A. Picardello, *The circle transform on trees*, Differential Geom. Appl. **19** (2003), no. 3, 295–305, DOI 10.1016/S0926-2245(03)00036-6.

Casadio_Tarabusi&Gindikin&Picardello [13] ———, *Integral geometry and harmonic analysis on homogeneous and semi-homogeneous treess.* in preparation.

llo-algebras_generated_by_Laplacians [14] E. Casadio Tarabusi and M. A. Picardello, *The algebras generated by the Laplace operators in a semi-homogeneous tree*, Trends in Harmonic Analysis, Springer INDAM Series, vol. 3, Springer-Verlag Italia, Milan Heidelberg New York Dordrecht London, 2013, pp. 77–90, DOI 10.1007/978-88-470-285-1_5.

[15] Cohen, J. M., Colonna, F., Gowrisankaran, K., and Singman, D.: Polyharmonic functions on trees. *Amer. J. Math.* **124** (2002) 999–1043.

Deligne [16] P. Deligne, *La conjecture de Weil, I*, Publications Mathématiques IHES, vol. 43, 1974, pp. 273–307.

Faraut&Picardello [17] J. Faraut and M. A. Picardello, *The Plancherel measure for symmetric graphs*, Ann. Mat. Pura Appl. **138** (1984), 151–155.

Figa-Talamanca&Nebbia [18] A. Figà-Talamanca and C. Nebbia, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees*, London Mathematical Society Lecture Notes Series, vol. 61, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, and Sydney, 1991.

Figa-Talamanca&Picardello-JFA [19] A. Figà-Talamanca and M. A. Picardello, *Spherical functions and harmonic analysis on free groups*, J. Funct. Anal. **47** (1982), 281–304.

Figa-Talamanca&Picardello [20] ———, *Harmonic Analysis on Free Groups*, Lecture Notes in Pure and Applied Mathematics, vol. 87, Marcel Dekker Inc., New York, 1983.

Figa-Talamanca&Picardello-PAMS [21] ———, *Restriction of Spherical Representations of $PGL_2(Q_p)$ to a Discrete Subgroup*, Proc. Amer. Math. Soc. **91** (1984), 405–408.

Figa-Talamanca&Steger [22] A. Figà-Talamanca and T. Steger, *Harmonic Analysis for Anisotropic Random Walks on Homogeneous Trees*, Mem. Amer. Math. Soc., vol. 110, Amer. Math. Soc., Providence, 1994.

Furstenberg [23] H. Furstenberg, *Random walks and discrete subgroups of Lie groups*, Advances Probab. and Related Topics, Vol. 1, M. Dekker, New York, 1971, pp. 1–63.

Gelfand&Graev [24] I. M. Gelfand and M. I. Graev, *Analogue of the Plancherel formula for the classical groups*, Trudy Moscow Mat. **4** (1955), 375–404.

Gelfand&Graev&Vilenkin [25] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions*, Vol. 5, Academic Press, New York, 1966.

Gelfand&Graev&Pyatetskii [26] I. M. Gel'fand, M. I. Graev, and N. Ya. Pyatetskii-Schapiro, *Generalized Functions*, Vol. 6: Representation Theory and Automorphic Functions, Academic Press, New York, 1969.

Haagerup [27] U. Haagerup, *An example of a non-nuclear C^*-algebra which has the metric approximation property*, Inventiones Math. **50** (1979), 279–293.

Ha [28] M. Hamana, *Injective envelopes of Banach modules*, Tohoku Math. Journal **30** (1978), 439–453.

Helgason-GGA [29] S. Helgason, *Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators and Symmetric Spaces*, Academic Press, New York, 1984.

Iozzi&Picardello [30] A. Iozzi and M. A. Picardello, *Graphs and convolution operators*, Topics in Modern Harmonic Analysis, Vol. 1, Ist. Naz. Alta Mat., Roma, 1983, pp. 187–209.

Iozzi&Picardello-Springer [31] ———, *Spherical functions on symmetric graphs*, Lecture Notes in Math., vol. 993, Springer-Verlag, New York-Berlin, 1983, pp. 344–386.

[32] Kashiwara, M., Kowata, A., Minemura, K., Okamoto, K., Oshima, T., and Tanaka, M.: Eigenfunctions of invariant differential operators on a symmetric space, *Ann. of Math.* **107** (1978), 1–39.

Kato_1 [33] S. Kato, *Irreducibiity of principal series representations for Hecke algebras of affine type*, J. Fac. Sci. Univ. Tokyo IA Math. **28** (1981), 929–943.

Kato_2 [34] ———, *On eigenspaces of the Hecke algebras with respect to a good maximal compact subgroup of a $p-$adic reductive group*, Math. Ann. **257** (1981), 1–7.

Macdonald [35] I. G. Macdonald, *Spherical functions on a group of p-adic type*, Ramanujan Institute, Centre for Advanced Study in Mathematics, University of Madras, Madras, 1971. Publications of the Ramanujan Institute, No. 2.

Mantero&Zappa [36] A. M. Mantero and A. Zappa, *The Poisson transform on free groups and uniformly bounded representations*, J. Funct. Anal. **51** (1983), 372–399.

Oz [37] N. Ozawa, *Boundaries of reduced free group $C*-$algebras*, Bull. London Math. Soc. **39** (2007), 35–38.

[38] M. A. Picardello, W. Woess, *Random walks on amalgams*, Monatshefte Math. **100** (1985), 21-33.

[39] M. A. Picardello, *Positive definite functions and L^p-convolution operators on amalgams*, Pacific J. Math. **123** (1986), 209-221.

[40] M. A. Picardello, W. Woess, *Martin boundaries of random walks: ends of trees and groups*, Trans. Amer. Math. Soc. **302** (1987), 285-305.

[41] A. Korányi, M. A. Picardello, M. H. Taibleson, *Hardy spaces on non–homogeneous trees*, Symp. Math. **29** (1987), 205-265.

[42] M. A. Picardello, P. Sjögren, *The minimal Martin boundary of a cartesian product of trees*, Proc. Centre Math. Anal. Austral. Nat. Univ. **16** (1988), 226-246.

[43] M. A. Picardello, W. Woess, *Harmonic functions and ends of graphs*, Proc. Edinburgh Math. Soc. **31** (1988), 457-461.

[44] M. A. Picardello, W. Woess, *Ends of graphs, potential theory and electric networks*, in "Cycles and Rays", NATO ASI Ser. C, Kluwer Academic Publishers, Dordrecht (1990), 181-196.

[45] M. A. Picardello, M. H. Taibleson, *Degeneracy of Hardy spaces on a two-sheeted graph: a sandwich of trees*, Ars Combinatoria **29B** (1990), 161–174.

[46] M. A. Picardello, P. Sjögren, *Boundary behaviour of eigenfunctions of the Laplacian in a bi-tree* , J. Reine Angew. Math. **424** (1992), 133–144.

[47] M. A. Picardello, M. H. Taibleson, W. Woess, *Harmonic measure on the planar Cantor set from the viewpoint of graph theory*, Discrete Math. **109** (1992), 193–202.

[48] M. A. Picardello, W. Woess, *The full Martin boundary of the bi-tree*, Ann. Prob. **22** (1994), 2203–2222.

Picardello&Woess-PotentialAnalysis [49] M. A. Picardello and W. Woess, *Boundary representations of λ-harmonic and polyharmonic functions on trees*, Potential Analysis (2018).

Picardello&Woess-London [50] ———, *Multiple boundary representations of λ-harmonic functions on trees* (2019).

Ronan [51] M. A. Ronan, *Lectures on Buildings, updated and revised*, University of Chicago Press, Chicago, 2009.

Serre [52] J. P. Serre, *Trees*, Graduate Texts in Math., Springer–Verlag, 2002.

Woess [53] W. Woess, *Denumerable Markov Chains. Generating functions, Boundary Theory, Random Walks on Trees*, European Math. Soc. Publishing House, 2009.

Paunescu&Radulescu [54] L. Paunescu and F. Radulescu, *A generalisation to Birkhoff –von Neumann theorem*, Adv. Math. **308,** (2017,) 836–858.

Radulescu-Theta [55] F. Radulescu, *Free group factors and Hecke operators. The varied landscape of operator theory*, Theta Ser. Adv. Math., vol. 17, Theta, Bucharest, 2014.

Radulescu-Uspekhi [56] ———, *Endomorphisms of spaces of virtual vectors fixed under the action of a discrete group*, Uspekhi Mat. Nauk **71** (2016), 121-178.

Radulescu-CAOT [57] ———, *On the countable measure-preserving relation induced on a homogeneous quotient by the action of a discrete group*, Complex Anal. Oper. Theory **9** (2015), 121-178.

Radulescu-NG [58] ———, *The Operator Algebra content of the Ramanujan-Petersson Problem, space*, J. Non-comm. Geom. (2019). in print.

Popa [59] A. Popa, *On the trace formula for Hecke operators on congruence subgroups, II*, Research in the Mathematical Sciences, vol. 5, Springer-Verlag, 2018.

Sally&Shalika [60] P. Sally and J. A. Shalika, *The Plancherel formula for $SL(2)$ over a local field*, Proc. Nat. Acad. Sciences U.S.A. **63** (1969), 661–667.

[61] Fidaleo F. *Harmonic analysis on perturbed Cayley trees*, J. Func. Anal., **261** (2011), 604–634.

[62] Fidaleo F. *Corrigendum to "Harmonic analysis on perturbed Cayley Trees"*, J. Func. Anal., **262** (2012), 4634–4637.

[63] Fidaleo F. *Harmonic analysis on perturbed Cayley trees II: the Bose–Einstein condensation*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **15** (2011), 603–634.

[64] Fidaleo F. *Harmonic analysis on inhomogeneous amenable networks and the Bose–Einstein condensation*, J. Stat. Phys., **160** (2015), 715-759.

[65] Fidaleo F., Guido D., Isola T. *Bose Einstein condensation on inhomogeneous graphs*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **14** (2011), 149–197.

Buchholz-JFA [66] D. Buchholz, *The resolvent algebra: ideals and dimension*, J. Funct. Anal. **266** (2014), 3286–3302.

Buchholz-CMPh [67] ———, *The resolvent algebra of non-relativistic Bose fields: observables, dynamics and states*, Comm. Math. Phys. **362** (2018), 949–981.

Silvestrini [68] P. Silvestrini, R. Russo, V. Corato, B. Ruggiero, C. Granata, S. Rombetto, M. Russo, M. Cirillo, A. Trombettoni, and P. Sodano, *Topology-induced critical current enhancement in Josephson networks*, Phys. Lett. A **370** (2007), 499–503.

Lorenzo [69] M. Lorenzo et al., *On Bose–Einstein condensation in Josephson junctions stargraph arrays*, Phys. Lett. A **378** (2014), 655–658.

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy
Email address: picard@mat.uniroma2.it

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy
Email address: fidaleo@mat.uniroma2.it

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy
Email address: radulesc@mat.uniroma2.it