Solution 3 by Radouan Boukharfane.

We have

$$\sum_{\text{cyclic}} \frac{2a}{2a+b+c} \le \sum_{\text{cyclic}} \frac{1}{2}\left(\frac{a}{a+b}+\frac{a}{a+c}\right) = \frac{3}{2} \le \sum_{\text{cyclic}} \frac{a}{b+c}.$$

The first inequality is the AM-GM inequality; the second is Nesbitt's inequality.

Solution 4 by Phil McCartney.

Without loss of generality, we may assume that $a+b+c=1$, so that, for example,

$$\frac{a}{2b+2c} - \frac{a}{2a+b+c} = \frac{a}{2-2a} - \frac{a}{1+a} = \frac{a}{2}\left(\frac{3a-1}{1-a^2}\right).$$

Thus the claimed inequality is equivalent to

$$\sum_{\text{cyclic}} g(a) \ge 0, \text{ where } g(x) = x\left(\frac{3x-1}{1-x^2}\right) \text{ for } 0 \le x < 1.$$

On that interval,

$$g''(x) = \frac{2\left(-x^3+9x^2-3x+3\right)}{\left(1-x^2\right)^3} > 0,$$

so that g is convex there. By Jensen's inequality,

$$\sum_{\text{cyclic}} g(a) \ge 3 \cdot g\left(\frac{a+b+c}{3}\right) = 3 \cdot g\left(\frac{1}{3}\right) = 0.$$

Editor's note: notice the following

$$-x^3+9x^2-3x+3 = \left(4\sqrt{2}-x+3\right)\left(3-2\sqrt{2}-x\right)^2 + 16\left(3-2\sqrt{2}\right) > 0.$$

3764. [2012 : 285, 286] *Proposed by D. M. Bătineţu-Giurgiu and N. Stanciu.*

Let $(a_n)_{n\ge 1}$ be a positive real sequence such that $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{n} = a \in \mathbb{R}^+$. Compute

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{n+1} - \frac{\sqrt[n]{a_n!}}{n}\right),$$

where $a_1! = a_1$ and $a_n! = a_n \cdot a_{n-1}!$ for $n > 1$.

Solved by A. Alt; D. Koukakis; P. Perfetti; D. Văcaru; and the proposer. One other solution arrived at the correct answer via a step that the author did not clarify and the editor was unable to justify. We present the solution by Paolo Perfetti and the proposer (done independently).

We exploit the Cesaro-Stolz Theorem, which states the following: let $\{a_n\}$ and $\{b_n\}$ be real sequences such that $\{b_n\}$ is strictly increasing and unbounded and $\lim_{n\to\infty}(a_{n+1}-a_n)/(b_{n+1}-b_n)=L$, then $\lim_{n\to\infty} a_n/b_n = L$. Applying this theorem, we find that

$$\lim_{n\to\infty}\frac{a_n}{n^2}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)^2-n^2}=\lim_{n\to\infty}\left(\frac{a_{n+1}-a_n}{n}\right)\cdot\left(\frac{n}{2n+1}\right)=\frac{a}{2}.$$

Observe that

$$\frac{\sqrt[n+1]{a_{n+1}!}}{n+1}-\frac{\sqrt[n]{a_n!}}{n}=n\cdot\frac{\sqrt[n]{a_n!}}{n^2}\cdot(q_n-1)=\frac{\sqrt[n]{a_n!}}{n^2}\cdot\frac{(q_n-1)}{\ln q_n}\cdot\ln(q_n^n),$$

where

$$q_n=\frac{\sqrt[n+1]{a_{n+1}!}}{n+1}\cdot\frac{n}{\sqrt[n]{a_n!}}.$$

By the equality of the limits in the ratio and root tests,

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n!}}{n^2}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n!}{n^{2n}}}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^2}\cdot\left(\frac{n}{n+1}\right)^{2n}=\frac{a}{2}\cdot\frac{1}{e^2}=\frac{a}{2e^2}.$$

Also

$$\lim_{n\to\infty}q_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{(n+1)^2}\right)\cdot\left(\frac{n^2}{\sqrt[n]{a_n!}}\right)\cdot\left(\frac{n+1}{n}\right)=1,$$

so that

$$\lim_{n\to\infty}\frac{q_n-1}{\ln q_n}=1.$$

Finally,

$$\begin{aligned}\lim_{n\to\infty}q_n^n&=\lim_{n\to\infty}\left(\frac{a_{n+1}!}{a_n!}\right)\cdot\left(\frac{1}{\sqrt[n+1]{a_{n+1}!}}\right)\cdot\left(\frac{n}{n+1}\right)^n\\&=\lim_{n\to\infty}\left(\frac{a_{n+1}}{(n+1)^2}\right)\cdot\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}!}}\right)\cdot\left(\frac{n}{n+1}\right)^n\\&=\frac{a}{2}\cdot\frac{2e^2}{a}\cdot\frac{1}{e}=e.\end{aligned}$$

It follows that the desired limit is equal to $a/2e^2$.

3765. [2012 : 285, 287] *Proposed by M. Bataille.*

Let ABC be a triangle with circumcircle Γ and orthocentre H and let the circle with diameter AH intersect Γ again at K. Prove that

(a) $KB\cdot HC=KC\cdot HB$.

(b) lines KB, HC meet on the circle tangent to Γ at K and passing through H.