The result holds for $n = N + 1$ and the induction is complete. Equality holds only when $n = 0$.

(**d**) $\sqrt{s^2 + s'^2} \le C/t$. For $T_n \le t \le T_{n+1}$, we have

$$\frac{1}{\sqrt{s^2+s'^2}} \ge \frac{1}{U_n} \ge \frac{1}{U_0} + \frac{n}{3} > \frac{1}{U_0} + \frac{T_n - T_0}{3\pi} \ge \frac{1}{U_0} + \frac{(t-\pi) - T_0}{3\pi} > \frac{t}{C}$$

for $C = 10$ and for sufficiently large t. Any choice of C with $C > 3\pi$ will work.

If $a \ge 0$ and $b \le 0$, then we obtain the same asymptotic result, and $C = 10$ again suffices.

Also solved by E. A. Herman, O. P. Lossers (Netherlands), R. Stong, D. B. Tyler, E. I. Verriest, TCDmath Problem Group (Ireland), and the proposer.

More Triangle Inequalities

11664 [2012, 699]. *Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ, and Darij Grinberg, Massachusetts Institute of Technology, Cambridge, MA.* Let a, b, and c be the side lengths of a triangle. Let s denote the semiperimeter, r the inradius, and R the circumradius of that triangle. Let $a' = s - a$, $b' = s - b$, and $c' = s - c$.
(**a**) Prove that $\frac{ar}{R} \le \sqrt{b'c'}$.
(**b**) Prove that

$$\frac{r(a+b+c)}{R}\left(1 + \frac{R-2r}{4R+r}\right) \le 2\left(\frac{b'c'}{a} + \frac{c'a'}{b} + \frac{a'b'}{c}\right).$$

Solution for (a) by Alper Ercan, Istanbul, Turkey. Recall these formulas, involving the area Δ of the triangle:

$$\Delta = rs = \frac{abc}{4R}, \qquad \Delta^2 = sa'b'c'.$$

The inequality to be proved becomes $4a'\sqrt{b'c'} \le (a'+b')(a'+c')$, because $bc = (a'+b')(a'+c')$. By the AM–GM inequality, $2\sqrt{a'b'} \le a' + b'$ and $2\sqrt{a'c'} \le a' + c'$. The desired inequality follows.

Solution for (b) by Paolo Perfetti, Università degli Studi di Roma "Tor Vergata", Rome, Italy. Recall

$$R = \frac{abc}{\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}},$$
$$r = \frac{abc}{4Rs} = \frac{\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}}{s}.$$

For convenience, write $x = a'$, $y = b'$, $z = c'$, and $D = (x+y)(y+z)(z+x)$. The inequality to be proved becomes

$$\frac{8(x+y+z)xyz}{D}\,\frac{5D - 4xyz}{4D + 4xyz} \le 2\left[\frac{xy}{x+y} + \frac{yz}{y+z} + \frac{zx}{z+x}\right].$$

Clearing denominators, we see that this is equivalent to

$$(xy)^3 + (yz)^3 + (zx)^3 + 3(xyz)^2 \ge x^3y^2z + y^3z^2x + z^3x^2y,$$

which, with $\alpha = xy$, $\beta = yz$, and $\gamma = zx$, becomes Schur's third-degree inequality $\alpha^3 + \beta^3 + \gamma^3 + 3\alpha\beta\gamma \ge \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$.

Editorial comment. Some solvers noted that part (a) is related to Problem 11306, this **Monthly 116** (2009) 88–89. Part (b) was also solved using various other geometric inequalities, such as Kooi's inequality.

Also solved by R. Boukharfane (Canada), M. Can, R. Chapman (U. K.), P. P. Dálylay (Hungary), J. Fabrykowski & T. Smotzer, O. Geupel (Germany), M. Goldenberg & M. Kaplan, W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), P. Nüesch (Switzerland), V. Pambuccian, N. Stanciu & Z. Zvonaru (Romania), R. Stong, M. Vowe (Switzerland), J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

Inequalities for Inner Product Space

11667 [2012, 700]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Dan Schwarz, Softwin Co., Bucharest, Romania.* Let f, g, and h be elements of an inner product space over $\mathbb{R}$, with $\langle f, g\rangle = 0$.

(**a**) Show that

$$\langle f, f\rangle\langle g, g\rangle\langle h, h\rangle^2 \geq 4\langle g, h\rangle^2\langle h, f\rangle^2.$$

(**b**) Show that

$$(\langle f, f\rangle\langle h, h\rangle)\langle h, f\rangle^2 + (\langle g, g\rangle\langle h, h\rangle)\langle g, h\rangle^2 \geq 4\langle g, h\rangle^2\langle h, f\rangle^2.$$

Solution I by Pál Péter Dálylay, Szeged, Hungary. If f, g, or h is zero, then the inequalities clearly hold. Since $\langle f, g\rangle = 0$, note that

$$e = \frac{\langle h, f\rangle}{\langle f, f\rangle} f + \frac{\langle h, g\rangle}{\langle g, g\rangle} g$$

is the orthogonal projection of h onto the space spanned by $\{f, g\}$, and therefore $\|h\|^2 = \|e\|^2 + \|h - e\|^2$. Thus, $\|h\|^2 \geq \|e\|^2 = \langle h, f\rangle^2/\|f\|^2 + \langle h, g\rangle^2/\|g\|^2 \geq 2\langle h, f\rangle\langle h, g\rangle/(\|f\| \cdot \|g\|)$. Squaring both sides gives (a). By the AM–GM inequality,

$$\begin{aligned}\langle f,f\rangle\langle h,h\rangle\langle h,f\rangle^2+\langle g,g\rangle\langle h,h\rangle\langle g,h\rangle^2 &\geq 2\sqrt{\langle f,f\rangle\langle g,g\rangle(\langle h,h\rangle)^2(\langle h,f\rangle)^2(\langle g,h\rangle)^2}\\ &= 2\cdot\left[\sqrt{\langle f,f\rangle\langle g,g\rangle}\langle h,h\rangle\right]\cdot\langle h,f\rangle\langle g,h\rangle.\end{aligned}$$

By (a), the bracketed term is at least $2|\langle g, h\rangle\langle h, f\rangle|$, so (b) follows.

Solution II by Paolo Perfetti, Dipartimento di Matematica, Università Degli Studi di Roma, Rome, Italy. If f, g, or h is zero, then the result clearly holds, so we may define $F = f/\|f\|$, $G = g/\|g\|$, and $H = h/\|h\|$. Now (a) reads $\langle h, h\rangle^2 \geq 4\langle G, h\rangle^2\langle F, h\rangle^2$. By Bessel's inequality, $\|h\|^2 \geq \langle G, h\rangle^2 + \langle F, h\rangle^2 \geq 2\langle G, h\rangle\langle F, h\rangle$, and (a) follows by squaring this result. Similarly, part (b) reads $\|f\|^2\langle H, f\rangle^2 + \|g\|^2\langle H, g\rangle^2 \geq 4\langle H, f\rangle^2\langle H, g\rangle^2$. Again by AM–GM, $\|f\|^2\langle H, f\rangle^2+\|g\|^2\langle H, g\rangle^2 \geq 2\|f\| \cdot \|g\| \cdot |\langle H, f\rangle\langle H, g\rangle|$, and it remains to show $\|f\| \cdot \|g\| \geq 2|\langle H, f\rangle\langle H, g\rangle|$. When we multiply both sides by $\frac{\|h\|}{\|f\|\cdot\|g\|}$, this becomes $\|h\| \geq 2|\langle h, F\rangle\langle h, G\rangle|$, which is essentially (a).

Also solved by K. Andersen (Canada), G. Apostolopoulos (Greece), R. Boukharfane (Canada), P. Bracken, R. Chapman (U. K.), A. Ercan (Turkey), D. Fleischman, C. Georghiou (Greece), O. Geupel (Germany), K. Hanes, E. A. Herman, F. Holland, B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), M. A. Prasad (India), N. C. Singer, R. Stong, R. Tauraso (Italy), T. Trif (Romania), D. B. Tyler, E. I. Verriest, J. Vinuesa (Spain), R. Wyant & T. Smotzer, GCHQ Problem Solving Group (U. K.), and the proposers.