Analogously,

$$\Delta_2 = \frac{r_b}{2R}\Delta, \quad \Delta_3 = \frac{r_c}{2R}\Delta,$$

hence the result.

3809. *Proposed by Michel Bataille.*

For positive real numbers x, y, let

$$G(x,y) = \sqrt{xy}, \quad A(x,y) = \frac{x+y}{2}, \quad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}\,.$$

Prove that

$$G(x^x, y^y) \geq (Q(x,y))^{A(x,y)}\,.$$

Solved by AN-anduud Problem Solving Group ; R. Boukharfane ; C. Curtis ; P. Deiermann and H. Wang ; O. Kouba ; K. W. Lau ; P. Perfetti ; D. Smith ; and the proposer. One incorrect solution was received. We present the solution by Paolo Perfetti.

The given inequality is equivalent to

$$x^{\frac{x}{2}} x^{\frac{y}{2}} \geq \left(\sqrt{\frac{x^2+y^2}{2}}\right)^{\frac{x+y}{2}} \iff x^{\frac{2x}{x+y}} y^{\frac{2y}{x+y}} \geq \frac{x^2+y^2}{2},$$

which upon being divided by x^2 becomes

$$\frac{y^{\frac{2y}{x+y}}}{x^{\frac{2y}{x+y}}} \geq \frac{1}{2}\left(1 + \left(\frac{y}{x}\right)^2\right). \tag{1}$$

Without loss of generality, we assume that $x \leq y$. Let $t = \frac{y}{x}$. Then $t \geq 1$, $\frac{2y}{x+y} = \frac{2t}{1+t}$ and (1) becomes

$$t^{\frac{2t}{1+t}} \geq \frac{1+t^2}{2} \iff \frac{2t}{1+t}\ln t \geq \ln\left(\frac{1+t^2}{2}\right). \tag{2}$$

To prove (2), let $f(t) = \frac{2t}{1+t}\ln t - \ln\frac{1+t^2}{2}$, $t \geq 1$. Then by routine calculations, we find :

$$f'(t) = 2\left(\frac{1-t^2+(1+t^2)\ln t}{(1+t)^2(1+t^2)}\right).$$

We claim that

$$1 - t^2 + (1+t^2)\ln t \geq 0 \quad \text{for all} \quad t \geq 1. \tag{3}$$

Let $h(t) = \ln t - \dfrac{t^2-1}{1+t^2} = \ln t - 1 + \dfrac{2}{1+t^2}$. Then

$$h'(t) = \frac{1}{t} - \frac{4t}{(1+t^2)^2} = \frac{(1-t^2)^2}{t(1+t^2)^2} \geq 0,$$

so $h(t)$ is an increasing function.

Since $h(1) = 0$, we have $h(t) \geq 0$, from which (3) follows. Hence, $f'(t) \geq 0$, which implies that $f(t)$ is an increasing function. Since $f(1) = 0$, we conclude that $f(t) \geq 0$ for all $t \geq 1$, which establishes (2) and completes the proof.

3810. *Proposed by Ovidiu Furdui.*

Let $k > 0$ be a positive real number. Find the value of

$$\int_0^1 \int_0^1 \left\{ \frac{x^k}{y} \right\} dxdy,$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a.

Solved by Š. Arslanagić; R. Boukharfane; C. Curtis; O. Geupel; R. I. Hess; O. Kouba; J. Ling; D. Stone and J. Hawkins; and the proposer. One incorrect solution was received, although the error was of a purely algebraic variety. We present two solutions.

Solution 1, by Oliver Geupel.

For $0 \leq x \leq 1$, let $f(x) = \int\limits_0^1 \left\{ \frac{x}{y} \right\} dy$. Let $y = \frac{x}{t}$, then $dy = -\frac{x}{t^2}\, dt$ and we obtain :

$$\begin{aligned} f(x) = x \int_x^\infty \frac{\{t\}}{t^2}\, dt &= x \left(\int_x^1 \frac{dt}{t} + \lim_{n\to\infty} \sum_{\ell=1}^n \int_\ell^{\ell+1} \frac{t-\ell}{t^2}\, dt \right) \\ &= x \left(-\log x + \lim_{n\to\infty} \sum_{\ell=1}^n \left(\log(\ell+1) - \log \ell + \frac{\ell}{\ell+1} - 1 \right) \right) \\ &= -x\log x + x \cdot \lim_{n\to\infty} \left(\log(n+1) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} \right) \right) \\ &= -x\log x + x(1-\gamma). \end{aligned}$$

Hence, $f(x^k) = x^k(1-\gamma) - kx^k \log x$.

Let I be the integral to be evaluated. Then we have :

$$\begin{aligned} I = \int_0^1 f(x^k)\, dx &= \frac{1-\gamma}{k+1} - k \int_0^1 x^k \log x\, dx \\ &= \frac{1-\gamma}{k+1} - k \cdot \lim_{u\to 0+} \int_u^1 x^k \log x\, dx \\ &= \frac{1-\gamma}{k+1} - k \cdot \lim_{u\to 0+} \left[\frac{1}{k+1} x^{k+1} \log x - \frac{x^{k+1}}{(k+1)^2} \right]_u^1 \\ &= \frac{1-\gamma}{k+1} + \frac{k}{(k+1)^2}, \end{aligned}$$