Solution to problem U614

Statement

Let $a_k > 0$, $k = 1, 2, \dots$ and $r, s > 0$. Prove that for $s \geq r$ if S_1 converges, also S_2 converges

$$S_1 = \sum_{k=3}^{\infty} \frac{1}{a_k(\ln(\ln a_k))^r}, \qquad S_2 = \sum_{k=3}^{\infty} \frac{1}{a_k(\ln(\ln k))^s}$$

Solution It suffices to prove the $s = r$ case.

Let $c_k > 0$ and $d_k > 0$ be two sequences such that $\sum c_k < +\infty$ and $\sum d_k = +\infty$. It follows $\limsup d_k/c_k = \infty$ because otherwise $d_k/c_k \leq A$ definitively for a certain positive number A and then $\sum d_k$ also would converge.

Let's apply this result to our case supposing by contradiction that

$$\sum_{k=3}^{\infty} \frac{1}{a_k(\ln(\ln k))^r} = \infty$$

It follows

$$\limsup \frac{(\ln(\ln a_k))^r}{(\ln(\ln k))^r} = \infty$$

Let $K \doteq \{k_i\}_{i=1}^{\infty}$ the largest subsequence of the naturals such

$$k \in K \implies (\ln(\ln a_k))^r \geq A(\ln(\ln k))^r, \qquad A > 1$$

$$\ln(\ln a_k) \geq A^{1/r} \ln(\ln k) \iff a_k \geq e^{(\ln k)^{A^{1/r}}}$$

$$\sum_{k=3, k\in K}^{\infty} \frac{1}{a_k(\ln(\ln k))^r} \leq \sum_{k=3, k\in K}^{\infty} \frac{1}{e^{(\ln k)^{A^{1/r}}} A(\ln(\ln k))^r} < \infty$$

because

$$\lim_{k\to\infty} \frac{k^{A^{1/r}}}{e^{(\ln k)^{A^{1/r}}}} = e^{A^{1/r}\ln k - (\ln k)^{A^{1/r}}} = e^{-\infty} = 0$$

Hence

$$\sum_{k=3, k\notin K}^{\infty} \frac{1}{a_k(\ln(\ln k))^r} = \infty$$

but this would mean

$$\limsup_{k\to\infty, k\notin K} \frac{(\ln(\ln a_k))^r}{(\ln(\ln k))^r} = \infty$$

contradicting that K is the largest set.

If $s < r$ the result is untrue. Let's take $a_k = k\ln k(\ln(\ln k))^{1-r+\delta}$, $\delta > 0$

$$\ln\ln a_k = \ln\ln(k\ln k(\ln(\ln k))^{1-r+\delta}) \geq \ln\ln k$$

hence

$$\sum_{k=3}^{\infty} \frac{1}{a_k(\ln(\ln a_k))^r} \leq \sum_{k=3}^{\infty} \frac{1}{k\ln k(\ln(\ln k))^{1-r+\delta}(\ln(\ln k))^r} =$$

$$= \sum_{k=3}^{\infty} \frac{1}{k\ln k(\ln(\ln k))^{1+\delta}} < \infty$$

This may be seen by the Cauchy–condensation–test

$$\sum_{k=3}^{\infty} \frac{1}{k\ln k(\ln(\ln k))^{1+\delta}} < \infty \iff \sum_{k=3}^{\infty} \frac{2^k}{2^k\ln(2^k)(\ln(\ln 2^k))^{1+\delta}} < \infty$$

$$\sum_{k=3}^{\infty} \frac{2^k}{2^k\ln(2^k)(\ln(\ln 2^k))^{1+\delta}} = \sum_{k=3}^{\infty} \frac{1}{k\ln 2(\ln k + \ln\ln 2))^{1+\delta}}$$

By applying Cauchy's test again

$$\sum_{k=3}^{\infty} \frac{1}{k\ln 2(\ln(k\ln 2))^{1+\delta}} < \infty \iff \sum_{k=3}^{\infty} \frac{2^k}{2^k\ln 2(\ln 2^k + \ln\ln 2))^{1+\delta}} < \infty$$

and this is true because the general term of the last series goes to zero as $k^{-1-\delta}$

On the other hand $\ln\ln a_k \leq C\ln\ln k$ for any $k \geq 3$ if C is large enough.

$$\sum_{k=3}^{\infty} \frac{1}{a_k(\ln(\ln k))^s} \geq \sum_{k=3}^{\infty} \frac{1}{k\ln k(\ln(\ln k))^{1-r+s+\delta}} = \infty$$

if $\delta < r - s$