We find that the y-coordinate of U is $\frac{m_2}{m_1 - m_2}$. We can thus calculate that the slope of UV is zero, which means that UV is parallel to l.

3867. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $(a_n)_{n\geq 1}$ be a positive real sequence and $a > 0$ such that

$$\lim_{n\to\infty} (a_n - a \cdot n!) = b > 0.$$

Find

$$\lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right).$$

We received four correct submissions and one incorrect solution. We present the solution of Paolo Perfetti, modified by the editor.

Note that $a_n = b + an! + o(1)$, and so

$$(a_n)^{\frac{1}{n}} = (b + an! + o(1))^{\frac{1}{n}} = (an!)^{\frac{1}{n}} \left(1 + \frac{b + o(1)}{an!}\right)^{\frac{1}{n}} \sim (an!)^{\frac{1}{n}}.$$

Using Stirling's formula, $n! = (n/e)^n \cdot \sqrt{2\pi n} \cdot (1 + o(1))$, we can write

$$\begin{aligned}(an!)^{\frac{1}{n}} &= a^{\frac{1}{n}} \cdot \frac{n}{e} (\sqrt{2\pi n})^{\frac{1}{n}} (1 + o(1))^{\frac{1}{n}} \\ &= \frac{n}{e} \cdot \exp\left(\frac{\ln(a \cdot \sqrt{2\pi n} \cdot (1 + o(1)))}{n}\right).\end{aligned}$$

Continuing from the last equality, the Taylor expansion for the exponential function then gives us

$$\begin{aligned}(an!)^{\frac{1}{n}} &= \frac{n}{e}\left(1 + \frac{\ln(a \cdot \sqrt{2\pi n} \cdot (1 + o(1)))}{n} + O\left(\frac{\ln^2 n}{n^2}\right)\right) \\ &= \frac{n}{e} + \frac{1}{e} \cdot \ln(a \cdot \sqrt{2\pi n} \cdot (1 + o(1))) + O\left(\frac{\ln^2 n}{n}\right).\end{aligned}$$

Hence

$$\begin{aligned}(a(n+1)!)^{\frac{1}{n+1}} - (an!)^{\frac{1}{n}} &= \tfrac{1}{e} + \tfrac{1}{e} \cdot \ln\left(\tfrac{a \cdot \sqrt{2\pi (n+1)} \cdot (1 + o(1))}{a \cdot \sqrt{2\pi n} \cdot (1 + o(1))}\right) + O\left(\tfrac{\ln^2 n}{n}\right) \\ &\to \tfrac{1}{e} \text{ as } n \to \infty.\end{aligned}$$

Since $\sqrt[n+1]{a_{n+1}} \sim (a(n+1)!)^{\frac{1}{n+1}}$ and $\sqrt[n]{a_n} \sim (an!)^{\frac{1}{n}}$ (see the beginning of the proof), it follows that $\lim\limits_{n\to\infty} \sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} = \frac{1}{e}$ as well.

3868. *Proposed by Iliya Bluskov.*

Determine the maximum value of $f(x, y, z) = xy + yz + zx - xyz$ subject to the constraint $x^2 + y^2 + z^2 + xyz = 4$, where x, y and z are real numbers in the interval $(0, 2)$.