3736. [2012 : 150, 151] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0, \infty) \to [0, \infty)$ be a bounded continuous function. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\int_0^\infty \ln^n(1 + e^{nf(x)})e^{-x}\, dx}\,.$$

Solution.

Let

$$I_n = \frac{1}{n}\left(\int_0^\infty \ln^n(1 + e^{nf(x)})e^{-x} dx\right)^{1/n}.$$

We show that $\lim_{n\to\infty} I_n = \|f\|_\infty \equiv \sup\{f(x) : x \geq 0\}$.

Observe that

$$nf(x) \leq \ln(1 + e^{nf(x)}) \leq \ln(2e^{nf(x)}) = \ln 2 + nf(x) \leq \ln 2 + n\|f(x)\|_\infty.$$

Therefore

$$I_n \leq \frac{1}{n} \left(\ln 2 + n\|f\|_\infty\right) \left(\int_0^\infty e^{-x} dx\right)^{1/n} = \frac{\ln 2}{n} + \|f\|_\infty,$$

whence $\limsup_{n\to\infty} I_n \leq \|f\|_\infty$.

On the other hand,

$$I_n \geq \frac{1}{n}\left(\int_0^\infty n^n f(x)^n e^{-x} dx\right)^{1/n} = \left(\int_0^\infty f(x)^n e^{-x} dx\right)^{1/n}.$$

Let $\epsilon > 0$ and select an interval $[a, b] \subseteq [0, \infty)$ for which $f(x) > \|f\|_\infty - \epsilon$ for $x \in [a, b]$. Then

$$\begin{aligned}\left(\int_0^\infty f(x)^n e^{-x} dx\right)^{1/n} &\geq \left(\int_a^b f(x)^n e^{-x} dx\right)^{1/n} \geq (\|f\|_\infty - \epsilon)\left(\int_a^b e^{-x} dx\right)^{1/n} \\ &= (\|f\|_\infty - \epsilon)[e^{-a} - e^{-b}]^{1/n},\end{aligned}$$

so that $\liminf_{n\to\infty} \left(\int_0^\infty f(x)^n e^{-x} dx\right)^{1/n} \geq \|f\|_\infty - \epsilon$ for each $\epsilon > 0$. It follows that $\liminf_{n\to\infty} I_n \geq \|f\|_\infty$ and the desired result follows.

The solution is based on those of the following solvers: MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer. Two additional solutions submitted were flawed.