Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; EDMUND SWYLAN, Riga, Latvia; and the proposer. Note that in Lemma 2, it may happen that the function u could have a discontinuity outside of the range of v.

3740. [2012 : 150, 152] *Proposed by Yunus Tuncbilek, Ataturk High School of Science, Istanbul, Turkey.*

Let R, r, r_a, r_b, r_c represent the circumradius, inradius and exradii, respectively, of ΔABC. Find the largest k that satisfies

$$r_a^2 + r_b^2 + r_c^2 + (1+4k)r^2 \geq (7+k)R^2.$$

Composite of solutions by Michel Bataille, Rouen, France; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; modified by the editor.

We claim that the largest such k is $k = 1$. For the proof, we show that the inequality holds for $k = 1$ and fails for all larger k. We use two known results:

$$r_a^2 + r_b^2 + r_c^2 = (4R+r)^2 - 2s^2, \tag{1}$$

where s is the semiperimeter, and

$$4R^2 + 4Rr + 3r^2 \geq s^2, \tag{2}$$

known as Gerretsen's inequality. [See [1] and [2].]

By (1), the inequality in the problem statement is equivalent to

$$(4R+r)^2 - 2s^2 + (1+4k)\,r^2 \geq (7+k)\,R^2,$$

and thus to

$$9R^2 + 8Rr + 2r^2 - 2s^2 \geq k\left(R^2 - 4r^2\right). \tag{3}$$

If $k = 1$, (3) is equivalent to

$$8R^2 + 8Rr + 6r^2 - 2s^2 \geq 0,$$

which is equivalent to (2). Hence the inequality holds for $k = 1$.

To show that there is no larger k for which the inequality always holds, let $t \in (0,1)$ and $a = t$, $b = c = 1$. The semiperimeter of this triangle is $s = \dfrac{2+t}{2}$, its area is $F = \dfrac{t}{4}\sqrt{4-t^2}$, its circumradius is $R = \dfrac{abc}{4F} = \dfrac{1}{\sqrt{4-t^2}}$, and its inradius is $r = \dfrac{F}{s} = \dfrac{t\sqrt{4-t^2}}{2\,(2+t)}$. For this triangle, therefore, inequality (3) is successively equivalent to

$$\frac{9}{4-t^2}+\frac{4t}{2+t}+\frac{t^2\left(4-t^2\right)}{2\left(2+t\right)^2}-\frac{\left(2+t\right)^2}{2}\geq k\left[\frac{1}{4-t^2}-\frac{t^2\left(4-t^2\right)}{\left(2+t\right)^2}\right]$$
$$\frac{\left(1+t\right)^2\left(1-t\right)^2}{\left(2+t\right)\left(2-t\right)}\geq k\cdot\frac{\left(1-t\right)^2\left(1+2t-t^2\right)}{\left(2+t\right)\left(2-t\right)}$$
$$\frac{\left(1+t\right)^2}{1+2t-t^2}\geq k. \qquad (4)$$

Fix $k>1$ and let $t\to 0$. Since $\dfrac{\left(1+t\right)^2}{1+2t-t^2}\to 1$, (4) eventually fails.

References

[1] O. Bottema, R.Z. Djordjević, R.R. Janic, D.S. Mitrinović, P.M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff Publishing, 1969, p. 50.

[2] K. W. Feuerbach, *Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks*, 1822, p. 5.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. Four incomplete solutions were received.