In other words, we have shown that if a face of our tetrahedron has a nonacute angle at one vertex, the face of the tetrahedron opposite that vertex must be an acute triangle. It follows that for our tetrahedron to have three nonacute face angles, they would necessarily have C as a vertex. But it is easily seen that the circumcentre of a tetrahedron having three nonacute angles at C would necessarily be exterior to the tetrahedron: Returning to the figure, we see that the line through C perpendicular to the plane BCD (and therefore parallel to OO_1) would intersect the sphere at a point, call it P that is separated from the vertex A by the plane through BD that is parallel to CP. The faces BCP and DCP of the tetrahedron $PBCD$ have right angles at P. To make those angles obtuse, P would have to be chosen on the sphere further from A; that is, at least one of the angles BCA and DCA would have to be acute.

3671★. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $ABCD$ be a tetrahedron and let M be a point in its interior. Prove or disprove that

$$\frac{[BCD]}{AM^2} = \frac{[ACD]}{BM^2} = \frac{[ABD]}{CM^2} = \frac{[ABC]}{DM^2} = \frac{2}{\sqrt{3}},$$

if and only if the tetrahedron is regular and M is its centroid. Here $[T]$ denotes the area of T.

No solutions have been received. This problem remains open.

3672. [2011 : 389, 392] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x and y be real numbers such that $x^2 + y^2 = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+xy} \ \geq \ \frac{3}{1+\left(\frac{x+y}{2}\right)^2}.$$

When does this inequality occur?

Solution by Arkady Alt, San Jose, CA, USA; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Marian Dincă, Bucharest, Romania; Dimitrios Koukakis, Kato Apostoloi, Greece; Kee-Wai Lau, Hong Kong, China; Salem Malikić, student, Simon Fraser University, Burnaby, BC; Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA; Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Cománeşti, Romania; and the proposer.

Let $t = xy$. Since $2|xy| \leq x^2 + y^2 = 1$, $|t| \leq \frac{1}{2}$. The difference between the two sides of the proposed inequality is

$$\begin{aligned}\frac{2+x^2+y^2}{1+x^2+y^2+x^2y^2}+\frac{1}{1+xy}-\frac{12}{4+x^2+y^2+2xy}\\ &= \frac{3}{2+t^2}+\frac{1}{1+t}-\frac{12}{5+2t}\\ &= \frac{(1-2t)(1+3t+5t^2)}{(2+t^2)(1+t)(5+2t)} = \frac{(1-2t)(1+t^2+(1+3t)^2)}{2(2+t^2)(1+t)(5+2t)}\\ &\geq 0\end{aligned}$$

with equality if and only if t $= 1/2$. With the given condition, this implies that equality occurs if and only if $x = y = \pm 1/\sqrt{2}$.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; STAN WAGON, Macalester College, St. Paul, MN, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA;

Wagon used mathematical software to find that, when $x^2 + y^2 = 1$,

$$3 \leq \left(\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{1}{1+xy}\right)\left(1+\left(\frac{x+y}{2}\right)^2\right) \leq \frac{10}{3}$$

with equality on the left if and only if $x = y = \pm 1/\sqrt{2}$. and equality on the right if and only if $x = -y = \pm 1/\sqrt{2}$.

3673. [2011 : 390, 392] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate the product

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)^{(-1)^{n-1}}.$$

I. Composite of solutions by Paul Bracken, University of Texas, Edinburg, TX, USA; Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA; Dimitrios Koukakis, Kato Apostoloi, Greece; and AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

The answer is $\pi^2/8$. Recall the Wallis formula:

$$\lim_{m\to\infty}\frac{1}{2m+1}\prod_{k=1}^{m}\frac{(2k)^2}{(2k-1)^2} = \frac{\pi}{2}.$$

For $n \geq 2$, let

$$P(n) = \prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)^{(-1)^{k-1}}.$$

Then

$$\begin{aligned}
P(2m) &= \prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)(2k+1)}\right]\prod_{k=1}^{m-1}\left[\frac{(2k)(2k+2)}{(2k+1)^2}\right] \\
&= \frac{(2m)^2}{(2m-1)(2m+1)}\prod_{k=1}^{m-1}\left[\frac{(2k)^2}{(2k+1)^2}\right]\prod_{k=1}^{m-1}\left[\frac{2k}{2k-1}\right]\prod_{k=1}^{m-1}\left[\frac{2k+2}{2k+1}\right] \\
&= \frac{(2m)^2}{(2m-1)(2m+1)}\left(\frac{2m}{2m-1}\right)\frac{1}{2}\frac{1}{(2m)^2}\prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)^2}\right] \\
&\qquad\qquad \times\prod_{k=1}^{m-1}\left[\frac{2k}{2k-1}\right]\prod_{k=1}^{m-1}\left[\frac{2k}{2k-1}\right] \\
&= \frac{(2m)^2}{(2m-1)(2m+1)}\left(\frac{2m}{2m-1}\right)\frac{1}{2}\frac{1}{(2m)^2}\left[\frac{(2m-1)^2}{(2m)^2}\right] \\
&\qquad\qquad \times\prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)^2}\right]\prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)^2}\right] \\
&= \frac{2m+1}{2(2m)}\left[\frac{1}{2m+1}\prod_{k=1}^{m}\frac{(2k)^2}{(2k-1)^2}\right]^2,
\end{aligned}$$

and

$$P(2m+1) = \frac{(2m)(2m+2)}{(2m+1)^2}P(m).$$

Hence

$$\begin{aligned}
\lim_{m\to\infty}P(2m+1) &= \lim_{m\to\infty}\frac{(2m)(2m+2)}{(2m+1)^2}\lim_{m\to\infty}P(2m) \\
&= \lim_{m\to\infty}P(2m) = \lim_{m\to\infty}\left[\frac{2m+1}{4m}\right]\frac{\pi^2}{4} = \frac{\pi^2}{8}.
\end{aligned}$$

II. Composite of solutions by Michel Bataille, Rouen, France; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

Recall Stirling's formula:

$$\lim_{n\to\infty}\frac{(e^n)n!}{n^n\sqrt{2\pi n}} = 1.$$

Defining $P(n)$ as in Solution I, we have that

$$\begin{aligned} P(2m+1) &= \frac{1}{2} \cdot \frac{[2\cdot 4\cdot 6\cdots(2m)]^4(2m+2)(2m+1)}{[3\cdot 5\cdot 7\cdots(2m+1)]^4} \\ &= \frac{[2\cdot 4\cdots 6\cdots(2m)]^8(m+1)(2m+1)}{[(2m+1)!]^4} \\ &= \frac{2^{8m}(m+1)(m!)^8}{(2m+1)^3[(2m)!]^4}. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{m\to\infty} P(2m+1) &= \lim_{m\to\infty} \frac{2^{8m}(m+1)(2\pi m)^4 m^{8m}e^{-8m}}{(2m+1)^3(4\pi m)^2(2m)^{8m}e^{-8m}} \\ &= \lim_{m\to\infty} \frac{(m+1)m^2\pi^2}{(2m+1)^3} = \frac{\pi^2}{8} \end{aligned}$$

and

$$\lim_{m\to\infty} P(2m) = \lim_{m\to\infty} \frac{(2m+1)^2}{(2m)(2m+2)} P(2m+1) = \frac{\pi^2}{8}.$$

III. Composite of solution by Richard I. Hess, Rancho Palos Verdes, CA, USA; Dimitrios Koukakis, Kato Apostoloi, Greece; Missouri State University Problem Solving Group, Springfield, MO; Skidmore College Problem Group, Saratoga Springs, NY; Albert Stadler, Herrliberg, Switzerland; and the proposer.

We use the infinite product representations for the sine and cosine functions:

$$\sin \pi x = \pi x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right);$$

$$\cos \pi x = (1-4x^2)\prod_{n=1}^{\infty}\left(1 - \frac{4x^2}{(2n+1)^2}\right).$$

The product of the problem can be written as a fraction whose numerator is

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{(2n+1)^2}\right) = \lim_{x\to\frac{1}{2}} \frac{\cos \pi x}{1-4x^2} = \lim_{x\to\frac{1}{2}} \frac{\pi \sin \pi x}{8x} = \frac{\pi}{4},$$

and whose denominator is

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{(2n)^2}\right) = \frac{\sin \pi/2}{\pi/2} = \frac{2}{\pi}.$$

It follows from this that the answer is $\pi^2/8$.

There were variants on the third solution. Stadler and the Skidmore group avoided consideration of the cosine product by writing the product as a fraction with numerator $\prod(1-1/n^2)$ *and denominator* $\prod(1-1/4n^2)^2$*. Koukakis used the Wallis formula instead of the sine product to calculate the denominator.*

Stan Wagon generalized the product to

$$\prod_{n=2}^{\infty}\left(1-\frac{a}{n^2}\right)^{(-1)^n},$$

so that $a = 1$ *gives the reciprocal of the given product. Using mathematical software, he then provided the answer for specific values of* a*. For example, when* $a = 2$*, the product is* $-\sqrt{2}\tan(\pi/\sqrt{2})/\pi$*; when* $a = 1/2$*, it is* $\sqrt{2}\tan(\pi/2\sqrt{2})/\pi$*; when* $a = 4$*, it is* 0 *and when* $a = 1/4$*, it is* $3/\pi$*. He concludes that, setting* $b = 1/a$*, "indeed, there seems to be general formula here that looks like*

$$\frac{2(b-1)\tan(\pi/2\sqrt{b})}{\pi\sqrt{b}},$$

though I have not investigated the exact range of truth for it." When $a = b = 1$*, we find through l'Hôpital's Rule that the limit as b tends to 1 is the expected* $8/\pi^2$*.*

Wagon's recourse to mathematical software raises two issues. Many of the Crux problems can be easily handled by machine, but are still worth posing when they can attract solutions that reveal underlying structure or when they draw attention to a particularly comely mathematical fact. The challenge is not to just solve the problem, but to do so in a way that is elegant, interesting or insightful. The efficiency of the software allows for experimentation that leads to conjectures not otherwise obtainable. This problem is a good example of these effects.

The proposer also asked for the values of the infinite products

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{(-1)^{n-1}}$$

and

$$\prod_{n=2}^{\infty}\left(\frac{n^2+1}{n^2-1}\right)^{(-1)^{n-1}}.$$

The value of the first is $\frac{\pi}{2}\tanh\frac{\pi}{2}$ *and of the second is* $\frac{2}{\pi}\tanh\frac{\pi}{2}$*.*

3674★. *Proposed by Zhang Yun, High School attached to Xi$^{/}$ An Jiao Tong University, Xi$^{/}$ An City, Shan Xi, China.*

Let I denote the centre of the inscribed sphere of a tetrahedron $ABCD$ and let A_1, B_1, C_1, D_1 denote their symmetric points of point I about planes BCD, ACD, ABD, ABC respectively. Must the four lines AA_1, BB_1, CC_1, DD_1 be concurrent?

No solutions have been received. This problem remains open.

3675. [2011 : 390, 392] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be the sides of a triangle and let s be its semiperimeter. Let r and R denote its inradius and circumradius respectively. Prove that

$$6 \;\le\; \sum_{\text{cyclic}}\frac{b(s-b)+c(s-c)}{a(s-a)} \;\le\; \frac{3R}{r}\,.$$

Solution by the proposer.