Part (a) was also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHARLES DIMINNIE and ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; EDMUND SWYLAN, Riga, Latvia; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Solvers used many strategies to narrow down the field, as exemplified in the featured solution. In particular, Hess and Yiu looked at the equation modulo 100. For $r \geq 2$, $(3n)^2 \equiv 33k - 24$*; it can be checked that the right member is a quadratic residue (modulo 100) only if* $k = 1$ *or* $k = 5$*. The case* $r = 2$ *is quickly disposed of. When* $r \geq 3$ *and* $k = 1$*, then* $3n^2 \equiv 111 \equiv 7 \pmod 8$*, which is impossible. There was one incomplete solution.*

Part (b) was solved only by Yiu.

3663. [2011 : 320, 322] *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Let a, b, c be positive real numbers. Prove that

$$\sqrt[3]{\frac{2a}{4a+4b+c}} + \sqrt[3]{\frac{2b}{4b+4c+a}} + \sqrt[3]{\frac{2c}{4c+4a+b}} < 2\,.$$

I. Solution by the proposer, modified slightly by the editor.

We first establish the following lemma :

Lemma 1 *If x and y are positive real numbers, then*

$$\sqrt[3]{4(x+y)} \geq \sqrt[3]{x} + \sqrt[3]{y}$$

with equality if and only if $x = y$.

Proof. Since $4(x^3+y^3) - (x+y)^3 = 3(x^3 - x^2y - xy^2 + y^3) = 3(x-y)(x^2-y^2) = 3(x-y)^2(x+y) \geq 0$, we have $4(x^3+y^3) \geq (x+y)^3$ or $\sqrt[3]{4(x^3+y^3)} \geq x+y$. Replacing x and y with $\sqrt[3]{x}$ and $\sqrt[3]{y}$, respectively, the lemma follows.

The given inequality is equivalent to

$$\sqrt[3]{\frac{a}{16a+16b+4c}} + \sqrt[3]{\frac{b}{16b+16c+4a}} + \sqrt[3]{\frac{c}{16c+16a+4b}} < 1. \qquad (1)$$

Using the lemma twice we have

$$\sqrt[3]{\frac{a}{16a+16b+4c}} = \sqrt[3]{\frac{a}{4((4a+4b)+c)}} \leq \frac{\sqrt[3]{a}}{\sqrt[3]{4(a+b)} + \sqrt[3]{c}} \leq \frac{\sqrt[3]{a}}{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}.$$

Hence,

$$\sum_{\text{cyclic}} \sqrt[3]{\frac{a}{16a+16b+4c}} \leq \sum_{\text{cyclic}} \frac{\sqrt[3]{a}}{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}} = 1.$$

If equality holds, then we must have $4a+4b = c$, $4b+4c = a$, and $4c+4a = b$ which imply that $8(a+b+c) = a+b+c$, so $a+b+c = 0$, a contradiction.

Hence (1) holds and our proof is complete.

II. Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, modified slightly by the editor.

We prove the stronger inequality that

$$\sum_{\text{cyclic}} \sqrt[3]{\frac{2a}{4a+4b+c}} \le 6^{\frac{1}{3}}.$$

Let S denote the summation on the left side of the given inequality. By the power-mean-inequality, we have, for $x_i \ge 0$, $i = 1, 2, 3$, that

$$\left(\frac{\sum_{i=1}^{3} x_i^{\frac{1}{3}}}{3}\right)^3 \le \left(\frac{\sum_{i=1}^{3} x_i^{\frac{1}{2}}}{3}\right)^2$$

or

$$\sum_{i=1}^{3} x_i^{\frac{1}{3}} \le 3^{\frac{1}{3}} \left(\sum_{i=1}^{3} x_i^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

which implies that

$$S \le 3^{\frac{1}{3}} \left(\sum_{\text{cyclic}} \sqrt{\frac{2a}{4a+4b+c}}\right)^{\frac{2}{3}} = 6^{\frac{1}{3}} \sum_{\text{cyclic}} \sqrt{\frac{a}{4a+4b+c}}\,.$$

We need to prove that

$$\sum_{\text{cyclic}} \sqrt[3]{\frac{a}{4a+4b+c}} \le 1. \tag{2}$$

Since the function $\sqrt{x}$ is concave on $[0, \infty)$, Jensen's Inequality implies that

$$\begin{aligned}
\sum_{\text{cyclic}} \sqrt{\frac{a}{4a+4b+c}} &= \sum_{\text{cyclic}} \frac{4a+4c+b}{9(a+b+c)} \sqrt{\frac{9^2 a(a+b+c)^2}{(4a+4b+c)(4a+4c+b)^2}} \\
&\le \sqrt{\sum_{\text{cyclic}} \frac{81a(4a+4c+b)(a+b+c)^2}{9(a+b+c)(4a+4b+c)(4a+4c+b)^2}} \\
&= \sqrt{\sum_{\text{cyclic}} \frac{9a(a+b+c)}{(4a+4b+c)(4a+4c+b)}}.
\end{aligned}$$

Hence (2) would follow if we show that

$$\sum_{\text{cyclic}} \frac{9a(a+b+c)}{(4a+4b+c)(4a+4c+b)} \le 1. \tag{3}$$

By homogeneity, we may assume that $a+b+c=1$. Then (3) reduces to

$$\sum_{\text{cyclic}} \frac{9a}{(4a-3b)(4-3c)} \le 1$$

which is equivalent, in succession, to

$$9\sum_{\text{cyclic}} a(4-3a) \le (4-3a)(4-3b)(4-3c)$$

$$36-27(a^2+b^2+c^2) \le 64-48+36(ab+bc+ca)-27abc$$

$$27(a^2+b^2+c^2)+36(ab+bc+ca) \ge 27abc+20. \tag{4}$$

We now define $x \ge 0$ by $3(ab+bc+ca)=1-x^2$. Then $x<1$, $ab+bc+ca \le \frac{1}{3}$ and $a^2+b^2+c^2 = 1-\frac{2}{3}(1-x^2) = \frac{1+2x^2}{3}$. We employ the following reslt in the paper "On a Class of Three Variable Inequalities" by Vo Quoc Ba Can in the book *Mathematical Reflection, the first two years*, by Titu Andreescu, XYZ press, p. 480

$$\frac{(1+x)^2}{27} \le abc \le \frac{(1-x)^2(1+2x)}{27}.$$

In particular, $27abc \le (1-x)^2(1+2x)$. Hence to prove (4) it suffices to show that

$$9(1+2x^2)+12(1-x^2) \ge (1-x)^2(1+2x)+20$$

which upon simplifications, reduces to $21+6x^2 \ge 21-3x^2+2x^3$ or $2x^3-9x^2 \le 0$ or $x^2(2x-9) \le 0$ which is clearly true and the proof is complete.

[*Ed : By examining the above proof for equality cases, it is easy to see that equality holds if and only if* $a=b=c$.]

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; ALBERT STADLER, Herrliberg, Switzerland; and HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA.

Stadler used Hölder's Inequality together with some result which appeared in Crux in the past. He also obtained the sharper upper bound and pointed out the equality case.

3664. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$|(1-a^2b)(1-b^2c)(1-c^2a)| \le 3|1-abc|.$$

[*Ed.* : The proposer's solution was flawed. STAN WAGON, Macalester College, St. Paul, MN, USA showed using Mathematica that the inequality is most likely true. Thus the problem remains open.]