3670. [2011 : 389, 391] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $n \geq 2$ be an integer. Calculate

$$\int_0^1 \int_0^1 \int_0^1 \frac{dxdydz}{x+y+z}.$$

I. Solution by Michel Bataille, Rouen, France ; Brian D. Beasley, Presbyterian College, Clinton, SC, USA ; Paul Bracken, University of Texas, Edinburg, TX, USA ; Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Oliver Geupel, Brühl, NRW, Germany ; Richard I. Hess, Rancho Palos Verdes, CA, USA ; Anastasios Kotronis, Athens, Greece ; Mitch Kovacs, St. Bonaventure University, St. Bonaventure, NY, USA ; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

The integral is equal to

$$\begin{aligned}
&\int_0^1 \int_0^1 [\ln(1+y+z) - \ln(y+z)]dydz \\
=&\int_0^1 [(1+y+z)\ln(1+y+z) - (y+z)\ln(y+z) - 1]_0^1 dz \\
=&\int_0^1 [(2+z)\ln(2+z) - 2(1+z)\ln(1+z) + z\ln z]dz \\
=&\Big[\frac{1}{2}(2+z)^2\ln(2+z) - \frac{1}{4}(2+z)^2 - (1+z)^2\ln(1+z) \\
&\qquad + \frac{1}{2}(1+z)^2 + \frac{1}{2}z^2\ln z - \frac{1}{4}z^2\Big]_0^1 \\
=&\frac{9}{2}\ln 3 - 6\ln 2 = \frac{3}{2}\ln\left(\frac{27}{16}\right) = \ln\left(\frac{81\sqrt{3}}{64}\right),
\end{aligned}$$

where we have used the fact that $\lim\limits_{\epsilon \to 0^+} \epsilon^2 \ln \epsilon = 0$.

II. Solution by Mohammed Aassila, Strasbourg, France.

We show that, when $n \geq 2$,

$$I_n \equiv \int_0^1 \int_0^1 \cdots \int_0^1 \frac{dx_1 dx_2 \cdots dx_n}{x_1 + x_2 + \cdots + x_n} = \frac{(-1)^n}{(n-1)!} \sum_{k=2}^{n} \binom{n}{k} (-1)^k k^{n-1} \ln k.$$

Observe that

$$\begin{aligned}
I_n &= \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^\infty \exp\{-t(x_1 + x_2 + \cdots + x_n)\} dt dx_1 dx_2 \cdots dx_n \\
&= \int_0^\infty \left(\int_0^1 \exp(-tx)dx\right)^n dt = \int_0^\infty \left(\frac{1 - e^{-t}}{t}\right)^n dt.
\end{aligned}$$

Let $0 \leq m \leq n-1$. Then $(1-e^{-t})^n = (t - \frac{1}{2}t^2 + \cdots)^n = t^n + \cdots$ so that $D_t^m(1-e^{-t})^n = n(n-1)\cdots(n-m+1)t^{n-m} + \cdots = O(t^{n-m})$. Also

$$D_t^m(1-e^{-t})^n = \sum_{k=0}^{n}(-1)^{m+k}\binom{n}{k}k^m e^{-kt}.$$

Integrating by parts n times and making the substitution $s = kt$, we find that

$$\begin{aligned}
I_n &= \frac{1}{(n-1)!}\int_0^\infty \left(\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{n-k-1}k^n e^{-kt}\right)\ln t dt \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k-1}k^n\int_0^\infty e^{-kt}\ln t dt \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^{n-1}\int_0^\infty e^{-s}(\ln k - \ln s)ds \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{n-k}k^{n-1}\ln k \\
&\qquad \times \int_0^\infty e^{-s}ds - \frac{1}{(n-1)!}\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{n-k}k^{n-1}\int_0^\infty e^{-s}\ln s ds.
\end{aligned}$$

The quantity $\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^{n-1}$ in the second term vanishes since it is the nth order difference at 0 of the polynomial x^{n-1}. The first two terms of the sum $\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^{(n-1)}\ln k$ also vanish, since $n \geq 2$ and $\lim\limits_{\epsilon\to 0^+}\epsilon^{n-1}\ln\epsilon = 0$. The desired result follows.

III. Solution by the Missouri State University Problem Solving Group, Springfield, MO, USA.

For $n \geq 1$ and $t > 0$, define

$$F_n(t) = \int_0^1\int_0^1\cdots\int_0^1 \frac{dx_1dx_2\cdots dx_n}{x_1+x_2+\cdots+x_n+t}.$$

We show that, for $n \geq 1$,

$$F_n(t) = \frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n-k)^{n-1}\ln(t+n-k).$$

Since $F_1(t) = \ln(t+1) - \ln t$, this holds for $n = 1$. Assume that it holds up

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to the index $n \geq 1$. Then $F_{n+1}(t)$ is given by

$$\begin{aligned}
&\int_0^1 F_n(x_{n+1}+t)dx_{n+1} \\
&\quad = \frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^1 (x_{n+1}+t+n-k)^{n-1}\ln(x_{n+1}+t-n-k)dx_{n+1} \\
&\quad = \frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\frac{(x_{n+1}+t+n-k)^n\ln(x_{n+1}+t+n-k)}{n}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. - \frac{(x_{n+1}+t+n-k)^n}{n^2}\right]_0^1 \\
&\quad = \frac{1}{n!}\left[\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n+1-k)^n\ln(t+n+1-k)\right. \\
&\qquad\qquad \left. - \sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n-k)^n\ln(t+n-k)\right] \\
&\qquad\qquad - \frac{1}{n(n!)}\left[\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n+1-k)^n - \sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n-k)^n\right] \\
&\quad = \frac{1}{n!}\left[\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n+1-k)^n\ln(t+n+1-k)\right. \\
&\qquad\qquad \left. + \sum_{k=1}^{n+1}(-1)^k\binom{n}{k-1}(t+n+1-k)^n\ln(t+n+1-k)\right] \\
&\qquad\qquad - \frac{1}{n(n!)}\left[\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n+1-k)^n\right. \\
&\qquad\qquad \left. + \sum_{k=1}^{n+1}(-1)^k\binom{n}{k-1}(t+n+1-k)^n\right] \\
&\quad = \frac{1}{n!}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(t+n+1-k)^n\ln(t+n+1-k) \\
&\qquad\qquad - \frac{1}{n(n!)}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(t+n+1-k)^n.
\end{aligned}$$

The latter sum, being the $(n+1)^{\text{th}}$ difference of the polynomial $(t+n+1-k)^n$ in k of degree n vanishes, and the desired result follows by induction.

When $n \geq 2$, $F_n(0)$ is defined and equal to $\lim\limits_{t\to 0^+} F_n(t)$ and we obtain the representation obtained in Solution 2.

One incorrect solution was received. The generalization to an $n-$fold integral was also established by ALBERT STADLER, Herrliberg, Switzerland; and the proposer. Using computer software, STAN WAGON, Macalester College, St. Paul, MN, USA; discovered that, when a, b, c

are positive,

$$\int_0^a \int_0^b \int_0^c \frac{1}{x+y+z} dxdydz$$
$$= \frac{1}{2}\left[-c^2 \ln\frac{c}{a+c} - a\left(c + a\ln\frac{a+c}{a}\right) + c(2b+c)\ln\frac{b+c}{a+b+c}\right.$$
$$+ b^2 \ln\frac{(a+b)(b+c)}{b(a+b+c)} + a\left(c + a\ln\left(1+\frac{c}{a+b}\right)\right)$$
$$+ 2c\left(-c\ln\frac{b+c}{c} + (a+c)\ln\left(1+\frac{b}{a+c}\right) + b\ln\left(1+\frac{a}{b+c}\right)\right)$$
$$\left.+2b\left(-b\ln\frac{b+c}{b} + (a+b)\ln\left(1+\frac{c}{a+b}\right) + c\ln\left(1+\frac{a}{b+c}\right)\right)\right].$$

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