Then $\angle QSA = \angle Q'QR = \frac{\pi}{2}$. Since O is the circumcenter of isosceles triangle $Q'QS$,

$$\angle SQQ' = 2\angle OQQ' = 2\angle OBQ' = \angle QBQ'. \tag{2}$$

Let M be midpoint of QQ'. Points Q, M, S, A lie on the circle with diameter QA, because $\angle QMA = \frac{\pi}{2} = \angle QSA$. Thus

$$\angle SQQ' = \angle SQM = \angle SAM. \tag{3}$$

Observe that $\sin\angle SAQ = \frac{SQ}{AQ} = \frac{Q'Q}{AQ} = \frac{2MQ}{AQ} = 2\sin\angle MAQ = 2\sin\angle OAQ$. From that we get

$$\angle SAQ = \arcsin(2\sin\angle OAQ). \tag{4}$$

From $\angle OAQ = \frac{\angle QAB - \angle QAC}{2}$ and equations (2) through (4),

$$\angle QBQ' = \angle SQQ' = \angle SAM = \angle SAQ - \angle OAQ,$$

which is equation (1), as claimed.

For the inequality on the right, simply note that we have proved that the middle difference is the maximum of $|\angle PCB - \angle PBC|$ over all points $P \in \ell$, while Problem 2255 established that this difference is at most $|\angle PAB - \angle PAC|$. This observation concludes the proof.

Also solved by the proposer; no solution was published before now.

For an alternative proof of the right inequality, let $x = |\angle PAB - \angle PAC|$, $0 \le x < \frac{\pi}{3}$. The inequality to prove reduces to $\arcsin\left(2\sin\frac{x}{2}\right) \le \frac{3x}{2}$, for $0 \le x < \frac{\pi}{3}$, which is an elementary exercise. It is interesting to note that according to the solution of Problem 2255, the inequality there, namely $|\angle PAB - \angle PAC| \ge |\angle PCB - \angle PBC|$, holds for all isosceles triangles ABC for which $\angle A \ge \frac{\pi}{3}$ (and $\angle B = \angle C \le \frac{\pi}{3}$), while the inequality fails for some positions of P in isosceles triangles with $\angle A < \frac{\pi}{3}$. Note that $\arcsin\left(2\sin\frac{x}{2}\right)$ is no longer real for $x > \frac{\pi}{3}$, so that there are positions of P for which the right inequality of the present problem fails for isosceles triangles with $\angle A > \frac{\pi}{3}$.

3641. [2011 : 234, 237] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $0 \le x_1, x_2, \ldots, x_n < \pi/2$ be real numbers. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\sec(x_k)\right)\left(1 - \left(\frac{1}{n}\sum_{k=1}^{n}\sin(x_k)\right)^2\right)^{1/2} \ge 1\,.$$

I. Composite of similar solutions by Arkady Alt, San Jose, CA, USA; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

Let $f(x) = \sec x$, $g(x) = \sin x$ and set $\bar{x} = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k$. Since $f''(x) = \dfrac{1+\sin^2 x}{\cos^3 x} > 0$ and $g''(x) = -\sin x < 0$ for $0 < x < \dfrac{\pi}{2}$, f is convex and g is concave on the interval $(0, 1)$.

Hence Jensen's Inequality ensures that

$$\frac{1}{n}\sum_{k=1}^{n}\sec x_k \geq \sec(\bar{x}) \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n}\sin x_k \leq \sin(\bar{x}).$$

Therefore we have

$$\begin{aligned}\left(\frac{1}{n}\sum_{k=1}^{n}\sec(x_k)\right)\left(1-\left(\frac{1}{n}\sum_{k=1}^{n}\sin(x_k)\right)^2\right)^{1/2} &\geq \sec(\bar{x})(1-\sin^2(\bar{x}))^{1/2}\\ &= \sec(\bar{x})\cos(\bar{x}) = 1.\end{aligned}$$

II. Composite of virtually identical solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Salem Malikić, student, Simon Fraser University, Burnaby, BC.

By Cauchy-Schwarz Inequality we have

$$n\left(\sum_{k=1}^{n}\sin^2(x_k)\right) = \left(\sum_{k=1}^{n}1^2\right)\left(\sum_{k=1}^{n}\sin^2(x_k)\right) \geq \left(\sum_{k=1}^{n}\sin(x_k)\right)^2$$

so

$$\left(\frac{1}{n}\sum_{k=1}^{n}\sin(x_k)\right)^2 \leq \frac{1}{n}\sum_{k=1}^{n}\sin^2(x_k).$$

Hence,

$$\begin{aligned}&\left(\frac{1}{n}\sum_{k=1}^{n}\sec(x_k)\right)\left(1-\left(\frac{1}{n}\sum_{k=1}^{n}\sin(x_k)\right)^2\right)^{1/2}\\ &\qquad \geq \left(\frac{1}{n}\sum_{k=1}^{n}\sec(x_k)\right)\left(1-\frac{1}{n}\sum_{k=1}^{n}\sin^2(x_k)\right)^{1/2}\\ &\qquad = \left(\frac{1}{n}\sum_{k=1}^{n}\sec(x_k)\right)\left(\frac{1}{n}\sum_{k=1}^{n}(1-\sin^2(x_k))\right)^{1/2}\\ &\qquad = \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\cos(x_k)}\right)\left(\frac{1}{n}\sum_{k=1}^{n}\cos^2(x_k)\right)^{1/2}\\ &\qquad \geq \left(\prod_{k=1}^{n}\frac{1}{\cos(x_k)}\right)^{1/n}\left(\left(\prod_{k=1}^{n}\cos^2(x_k)\right)^{1/n}\right)^{1/2} = 1\end{aligned}$$

by the AM-GM Inequality.

Clearly, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.