[2] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; and the proposer.

3636. [2011 : 172, 174] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers such that $a+b+c+d=2$. Prove that

$$ab(a^2+b^2+c^2)+bc(b^2+c^2+d^2)+cd(c^2+d^2+a^2)+da(d^2+a^2+b^2) \;\le\; 2\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $x=a+c$ and $y=b+d$. Then $x+y=2$, and we have

$$\begin{aligned}\sum_{\text{cyclic}} ab(a^2+b^2+c^2) &\le (ab+bc+cd+da)(a^2+b^2+c^2+d^2+2ac+2bd)\\ &= (a+c)(b+d)\left((a+c)^2+(b+d)^2\right) = x^3y+xy^3\\ &= \frac{1}{8}\left((x+y)^4-(x-y)^4\right) \le \frac{1}{8}(x+y)^4 = 2.\end{aligned}$$

The example $(a,b,c,d)=(1,1,0,0)$ shows that the inequality is sharp.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

From the proof featured above it is easy to see that equality holds if and only if $(a,b,c,d)=(1,1,0,0)$ *or* $(0,1,1,0)$ *or* $(0,0,1,1)$ *or* $(1,0,0,1)$*. This was explicitly pointed out by AN-anduud problem solving group and Arslanagić.*

3637. [2011: 172, 174] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let x be a real number with $|x|<1$. Determine

$$\sum_{n=1}^{\infty}(-1)^{n-1}n\left(\ln(1-x)+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n}\right)\,.$$

I. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Observe that

$$\begin{aligned} S'(x) &= \sum_{n=1}^{\infty}(-1)^{n-1}n\left(\frac{-1}{1-x}+1+x+x^2+\cdots+x^{n-1}\right) \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}n\left(\frac{-1}{1-x}+\frac{1-x^n}{1-x}\right) = \frac{-x}{1-x}\sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1} \\ &= \frac{-x}{(1-x)(1+x)^2} = \frac{1}{4}\left[\frac{2}{(1+x)^2}-\frac{1}{1+x}-\frac{1}{1-x}\right]. \end{aligned}$$

Noting that $S(0)=0$, we deduce that

$$\begin{aligned} S(x) &= \frac{1}{4}\left[2-\frac{2}{1+x}-\ln(1+x)+\ln(1-x)\right] \\ &= \frac{1}{4}\left[\frac{2x}{1+x}+\ln\frac{1-x}{1+x}\right] = \frac{x}{2(1+x)}+\frac{1}{4}\ln\frac{1-x}{1+x}. \end{aligned}$$

Editor's comment: We can also write

$$S'(x) = \frac{1}{2}\left[\frac{x}{(1+x)^2}-\frac{1}{1-x^2}\right],$$

which leads to $S(x)=\frac{1}{2}[x(1+x)^{-1}-\tanh^{-1}x]$.

II. Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

$$\begin{aligned} S(x) &= -\sum_{n=1}^{\infty}(-1)^{n-1}n\left(\sum_{k=n+1}^{\infty}\frac{x^k}{k}\right) \\ &= -\sum_{n=2}^{\infty}\frac{x^n}{n}\sum_{j=1}^{n-1}(-1)^{j-1}j = -\sum_{n=2}^{\infty}\frac{x^n}{n}(-1)^n\left\lfloor\frac{n}{2}\right\rfloor \\ &= \sum_{n=2}^{\infty}(-1)^{n+1}\left\lfloor\frac{n}{2}\right\rfloor\frac{x^n}{n} = -\frac{1}{2}\sum_{k=1}^{\infty}x^{2k}+\sum_{k=1}^{\infty}\frac{kx^{2k+1}}{2k+1} \\ &= -\frac{1}{2}\sum_{k=1}^{\infty}x^{2k}+\frac{1}{2}\sum_{k=1}^{\infty}x^{2k+1}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{2k+1}}{2k+1} \\ &= -\frac{1}{2}\frac{x^2(1-x)}{1-x^2}-\frac{1}{4}\ln\frac{1+x}{1-x}+\frac{x}{2} \\ &= \frac{1}{2}\left[\frac{-x^2}{1+x}+x\right]+\frac{1}{4}\ln\frac{1-x}{1+x} = \frac{x}{2(1+x)}+\frac{1}{4}\ln\frac{1-x}{1+x}. \end{aligned}$$

III. Solution by Michel Bataille, Rouen, France.

For $|x| < 1$, we use the Taylor representation

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x-t)^n dt$$

with $f(x) = \ln(1-x)$ to obtain

$$\begin{aligned} \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} + \int_0^x \frac{(x-t)^n}{n!} \cdot \frac{-n!}{(1-t)^{n+1}} dt \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \int_0^x \frac{u^n}{1-u} du, \end{aligned}$$

where the two integrals are related by the substitution $(u-1)t = u - x$. Therefore

$$\begin{aligned} S(x) &= \sum_{n=1}^{\infty} \int_0^x \frac{(-1)^n n u^n}{1-u}\, du = \int_0^x \frac{1}{1-u} \left(\sum_{n=1}^{\infty} (-1)^n n u^n \right) du \\ &= \int_0^x \frac{-u}{(1-u)(1+u)^2}\, du = \frac{1}{4} \int_0^x \left[\frac{2}{(1+u)^2} - \frac{1}{1-u} - \frac{1}{1+u} \right] du \\ &= \frac{x}{2(1+x)} + \frac{1}{4} \ln \frac{1-x}{1+x}. \end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ANASTASIOS KOTRONIS, Athens, Greece; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3638. [2011: 234, 237] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle and let points D, E, F lie on lines BC, CA, AB, respectively, such that

$$BD : DC = \lambda : 1 - \lambda, \quad CE : EA = \mu : 1 - \mu, \quad AF : FB = \nu : 1 - \nu.$$

Show that DEF is a pedal triangle with regard to ΔABC if and only if

$$(2\lambda - 1)BC^2 + (2\mu - 1)CA^2 + (2\nu - 1)AB^2 = 0.$$

Solution by the proposer.

Let A', B', and C' be the midpoints of BC, CA, and AB, respectively. Since $\overrightarrow{BD} = \lambda \overrightarrow{BC}$ and $\overrightarrow{CD} = (\lambda - 1)\overrightarrow{BC}$, we have $(2\lambda - 1)\overrightarrow{BC} = 2\overrightarrow{A'D}$. Similarly, $(2\mu - 1)\overrightarrow{CA} = 2\overrightarrow{B'E}$ and $(2\upsilon - 1)\overrightarrow{AB} = 2\overrightarrow{C'F}$ so that the given condition is equivalent to

$$\overrightarrow{A'D} \cdot \overrightarrow{BC} + \overrightarrow{B'E} \cdot \overrightarrow{CA} + \overrightarrow{C'F} \cdot \overrightarrow{AB} = 0. \qquad (1)$$