It therefore suffices to prove that $|\lambda| = \frac{\beta}{\alpha}$. Because the M_i are concyclic, we deduce that ΔM_3MM_4 and ΔM_2MM_1 are inversely similar, as are ΔM_1MM_4 and ΔM_2MM_3. Consequently,

$$\frac{M_2M_1}{M_3M_4} = \frac{MM_1}{MM_4} = \frac{MM_2}{MM_3} \quad \text{and} \quad \frac{M_1M_4}{M_2M_3} = \frac{MM_1}{MM_2} = \frac{MM_4}{MM_3}.$$

As a result we have

$$\left(\frac{M_2M_1}{M_3M_4} \cdot \frac{M_1M_4}{M_2M_3}\right)^2 = \frac{MM_1}{MM_4} \cdot \frac{MM_2}{MM_3} \cdot \frac{MM_1}{MM_2} \cdot \frac{MM_4}{MM_3} = \left(\frac{MM_1}{MM_3}\right)^2;$$

that is, $\left(\frac{\beta}{\alpha}\right)^2 = \lambda^2$, and the desired equality $|\lambda| = \frac{\beta}{\alpha}$ follows.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany.

3624. [2011 : 114, 116] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n}\right).$$

I. Solution based on the approach of AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Let $a_0 = 0$, and, for $n \geq 1$, let

$$a_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n},$$

$$S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} a_k = \sum_{k=1}^{n} (a_k - a_{k-1}) a_k,$$

and

$$T_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} a_{k-1} = \sum_{k=1}^{n} (a_k - a_{k-1}) a_{k-1}.$$

Then

$$S_n + T_n = \sum_{k=1}^{n} (a_k - a_{k-1})(a_k + a_{k-1}) = \sum_{k=1}^{n} (a_k^2 - a_{k-1}^2) = a_n^2,$$

and

$$S_n - T_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} (a_k - a_{k-1}) = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Therefore $S_n = \frac{1}{2}\left[a_n^2 + \sum_{k=1}^{n} \frac{1}{k^2}\right]$. The desired sum is equal to

$$\lim_{n\to\infty} S_n = \frac{1}{2}\left[(\log 2)^2 + \frac{\pi^2}{6}\right].$$

II. Solution following approach of Richard I. Hess, Rancho Palos Verdes, CA, USA; Kee-Wai Lau, Hong Kong, China; the Missouri State University Problem Solving Group, Springfield, MO; and the proposer.

For positive integer m, let

$$S_m = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = \sum_{1 \le k \le n \le m} \frac{(-1)^{n-1}}{n} \frac{(-1)^{k-1}}{k}.$$

Interchanging the order of summation and relabeling the indices yields

$$\begin{aligned} S_m &= \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{n=k}^{m} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=n}^{m} \frac{(-1)^{k-1}}{k} \\ &= \sum_{n=1}^{m} \frac{1}{n^2} + \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=n+1}^{m} \frac{(-1)^{k-1}}{k}. \end{aligned}$$

Adding the two expressions for S_m yields that

$$S_m = \frac{1}{2} \left[\sum_{n=1}^{m} \frac{1}{n^2} + \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \right].$$

The required sum is

$$\lim_{m \to \infty} S_m = \frac{\pi^2}{12} + \frac{(\log 2)^2}{2}.$$

III. Solution by Oliver Geupel, Brühl, NRW, Germany(abridged).

When $a_n = \sum_{k=1}^{n} (-1)^{k-1}/k$, $b_n = \sum_{k=1}^{n} k^{-2}$ and $c_n = \sum_{k=1}^{n} (-1)^{k-1} a_k/k$, it can be proved by induction that

$$c_n = \frac{1}{2}(a_n^2 + b_n).$$

The required sum is equal to

$$\lim_{n \to \infty} c_n = \frac{1}{2}(\log 2)^2 + \frac{1}{12}\pi^2.$$

IV. Solution based on those of Anastasios Kotronονis, Athens, Greece; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and Albert Stadler, Herrliberg, Switzerland.

Since

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = \int_0^1 \sum_{k=1}^{n} (-x)^{k-1} dx = \int_0^1 \frac{1 - (-x)^n}{1 + x}\, dx,$$

the proposed sum is equal to

$$\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1\frac{1}{1+x}dx-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1\frac{(-x)^n}{1+x}\,dx$$
$$=(\log 2)^2-\int_0^1\frac{1}{1+x}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-x)^n}{n}\,dx$$
$$=(\log 2)^2-\int_0^1\frac{\log(1-x)}{1+x}\,dx$$
$$=(\log 2)^2-\left[\frac{(\log 2)^2}{2}-\frac{\pi^2}{12}\right]=\frac{(\log 2)^2}{2}+\frac{\pi^2}{12}.$$

No other solutions were received.

Perfetti provided a justification for the interchange of summation and integration in (IV), while Kotronis gave this determination of the final integral:

$$-\int_0^1\frac{\log(1-t)}{1+t}\,dt=\int_0^1\int_{-1}^0\frac{1}{1+t}\cdot\frac{t}{1+yt}\,dydt=\int_{-1}^0\int_0^1\frac{t}{(1+t)(1+yt)}\,dtdy$$
$$=\int_{-1}^0\int_0^1\left[\frac{1}{(y-1)(1+t)}-\frac{1}{(y-1)(1+yt)}\right]dtdy$$
$$=\int_{-1}^0\left[\frac{\log 2}{y-1}-\frac{\log(1+y)}{y(y-1)}\right]dy=-(\log 2)^2+\int_{-1}^0\left[\frac{\log(1+y)}{y}-\frac{\log(1+y)}{y-1}\right]dy$$
$$=-(\log 2)^2-\int_0^1\frac{\log(1-x)}{x}\,dx+\int_0^1\frac{\log(1-x)}{1+x}\,dx,$$

so that

$$\int_0^1\frac{\log(1-t)}{1+t}\,dt=\frac{(\log 2)^2}{2}+\frac{1}{2}\int_0^1\frac{\log(1-x)}{x}\,dx=\frac{(\log 2)^2}{2}-\frac{1}{2}\int_0^1\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}\,dx$$
$$=\frac{(\log 2)^2}{2}-\frac{1}{2}\sum_{n=1}^{\infty}\int_0^1\frac{x^{n-1}}{n}\,dx=\frac{(\log 2)^2}{2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{(\log 2)^2}{2}-\frac{\pi^2}{12}.$$

3625. [2011 : 114, 116] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\ \le\ 2\sqrt{1+\frac{abc}{(a+b)(b+c)(c+a)}}\,.$$

Solution by George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl,