3617. [2011 : 112, 115] *Proposed by Michel Bataille, Rouen, France.*

Let r be a positive rational number. Show that if r^r is rational, then r is an integer.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $a, b, c,$, and d be positive integers such that

$$r = \frac{a}{b},\ r^r = \frac{c}{d},\ gcd(a,b) = 1.$$

It follows that

$$\frac{a^a}{b^a} = r^a = r^{rb} = \frac{c^b}{d^b}.$$

We have to prove that $b = 1$. Assume the contrary. Then, b has a prime divisor p. Let $\varepsilon(n)$ denote the exact exponent of p in the integer n, that is, the non-negative integer α such that $p^\alpha \mid n$ and $p^{\alpha+1} \nmid n$. We obtain

$$a \cdot \varepsilon(b) + b \cdot \varepsilon(c) = \varepsilon(b^a c^b) = \varepsilon(a^a d^b) = a \cdot \varepsilon(a) + b \cdot \varepsilon(d) = b \cdot \varepsilon(d).$$

Hence, $a \cdot \varepsilon(b) = b \cdot (\varepsilon(d) - \varepsilon(c))$, from which we deduce that b divides the positive number $\varepsilon(b)$. Consequently, $b < p^b \le p^{\varepsilon(b)} \le b$. This is a contradiction, which completes the proof.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; KAYLIN EVERETT, California State University, Fresno, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3618. [2011 : 113, 115] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $\alpha > 3$ be a real number. Find the value of

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n}{(n+m)^\alpha}.$$

Composite of nearly identical solutions by Anastasios Kotronis, Athens, Greece; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

The original sum may be rewritten as

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n}{(n+m)^\alpha} &= \sum_{q=2}^{\infty}\sum_{p=1}^{q-1} \frac{p}{q^\alpha} = \sum_{q=2}^{\infty} \frac{1}{2} \cdot \frac{q(q-1)}{q^\alpha} = \frac{1}{2}\left[\sum_{q=2}^{\infty} \frac{1}{q^{\alpha-2}} - \sum_{q=2}^{\infty} \frac{1}{q^{\alpha-1}}\right] \\ &= \frac{1}{2}[(\zeta(\alpha-2)-1) - (\zeta(\alpha-1)-1)] \\ &= \frac{1}{2}(\zeta(\alpha-2) - \zeta(\alpha-1) - 1),\end{aligned}$$

where $\zeta(z)$ is the Riemann zeta function.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; ALBERT STADLER, Herrliberg, Switzerland; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer.

Stadler and AN-anduud Problem Solving Group used symmetry to write

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n}{(n+m)^{\alpha}}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{(n+m)^{\alpha}}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(n+m)-m}{(n+m)^{\alpha}}=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+m)^{\alpha-1}},$$

from which they proceeded in a similar fashion to the featured solution. The proposer proved

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+m)^{p}}=\zeta(p-1)-\zeta(p),$$

from which he used a symmetry argument similar to Stadler and AN-anduud's to set up a situation where the lemma could be used. Perfetti also supplied a proof of the convergence of the double sum. Wagon pointed out that Mathematica produces the desired result.

3619. [2011 : 113, 115] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$(a^2b-c)(b^2c-a)(c^2a-b)\ \le\ 4(ab+bc+ca-3a^2b^2c^2)\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

First we show that

$$a^2b+b^2c+c^2a+abc\le 4. \tag{1}$$

By the cyclicity of (1), we may without loss of generality suppose that either $a\le b\le c$ or $a\ge b\ge c$. In either case, the Rearrangement Inequality yields

$$a^2b+b^2c+c^2a+abc=a\cdot ab+b\cdot bc+c\cdot ac+b\cdot ac\le a\cdot ab+b\cdot ac+b\cdot ac+c\cdot bc=b(a+c)^2.$$

By the AM-GM Inequality, we have

$$b(a+c)^2=\frac{1}{2}\cdot 2b(a+c)(a+c)\le\frac{1}{2}\left(\frac{2b+(a+c)+(a+c)}{3}\right)^3=4,$$

which completes the proof of (1).

As a consequence of (1),

$$\sum_{\text{cyclic}}a^3b^2+\sum_{\text{cyclic}}(a^3bc+2a^2b^2c)=(a^2b+b^2c+c^2a+abc)(ab+bc+ca)\le 4(ab+bc+ca).$$

Moreover, by the AM-GM Inequality, we have

$$1=\left(\frac{a+b+c}{3}\right)^3\ge abc\ge(abc)^3$$

and

$$12(abc)^2\le 12(abc)^{11/6}\le\sum_{\text{cyclic}}(a^4bc^2+a^3bc+2a^2b^2c).$$