Solution to problem S613

Solve the equation

$$2(\sin x+\cos x)+\sec x+\csc x=4\sqrt{2}$$

Proof Evidently $x\neq k\pi$, $x\neq \pi/2+k\pi$, $k\in\mathbb{Z}$. The equation is

$$2\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right]+\frac{\sqrt{2}}{\sin x\cos x}\left[\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right]=4\sqrt{2}$$

$$\iff\quad 2\sin(x+\frac{\pi}{4})+\frac{\sin(x+\frac{\pi}{4})}{\sin x\cos x}=4$$

$$\iff\quad \sin(x+\frac{\pi}{4})(1+\sin(2x))=2\sin(2x)\ \underbrace{\iff}_{y=x+\pi/4}\ \sin y=\frac{-2\cos(2y)}{2\sin^2 y}$$

$$\iff\quad \sin^3 y-2\sin^2 y+1=(\sin y-1)(\sin^2 y-\sin y-1)=0$$

$$\sin y=1\quad\iff\quad y=\frac{\pi}{2}+2k\pi\ \implies\ x=\frac{\pi}{4}+2k\pi$$

$$\sin^2 y-\sin y-1=0\quad\iff\quad \sin y=\frac{1\pm\sqrt{5}}{2}\ \implies\ \sin y=\frac{1-\sqrt{5}}{2}$$

$$y=\arcsin\frac{1-\sqrt{5}}{2},\quad y=\pi-\arcsin\frac{1-\sqrt{5}}{2}$$

hence

$$x=-\arcsin\frac{\sqrt{5}-1}{2}-\frac{\pi}{4},\quad x=\frac{3\pi}{4}+\arcsin\frac{\sqrt{5}-1}{2}$$

Thus we have

$$x_1=\frac{\pi}{4}+2k,\quad x_2=\frac{3\pi}{4}+\arcsin\frac{\sqrt{5}-1}{2}+2k\pi,\quad x_3=\frac{\pi}{2}-x_2$$