Solution to problem O615

Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$abc(\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}) \leq 3$$

Proof

$$3\frac{1}{3}(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}) \leq 3\sqrt{\frac{a^2+b^2+c^2}{3}}$$

thus we prove

$$a+b+c=3 \implies (abc)^2(a^2+b^2+c^2) \leq 3$$

Moreover by $abc \leq (a+b+c)^3/27 \leq 1$ we come to prove

$$a+b+c=3 \implies abc(a^2+b^2+c^2) \leq 3$$

Let's change variables $a+b+c=3$, $ab+bc+ca=3v^2$, $abc=w^3$. The inequality reads as

$$u=1 \implies w^3(9u^2-6v^2) \leq 3$$

that is

$$f(w^3) \doteq w^3(9-6v^2) \leq 3 \quad (1)$$

The function $f(w^3)$ is linear increasing thus it holds if and only if it holds true for the minimum value of w. The minimum value of w is attained when $c=0$ (or cyclic) or $b=c$ (or cyclic).

$c=0$ is forbidden by the hypotheses but if we let $a, b, c,$ assume also that value we can observe that $w=0$ and the inequality clearly holds true.

If $c=b$ whence $a=(3-b)/2$ we have that $abc(a^2+b^2+c^2) \leq 3$ is equivalent to

$$\frac{3}{8}(a^3-6a^2+11a-8)(a-1)^2 \leq 3 \iff h(a) \doteq a^3-6a^2+11a-8 \leq 0 \quad 0 \leq a \leq 3$$

$$h'(a) = (a-2-\frac{1}{\sqrt{3}})(a-2+\frac{1}{\sqrt{3}}) \geq 0 \iff 0 \leq a \leq 2-\frac{1}{\sqrt{3}},\ 2+\frac{1}{\sqrt{3}} \leq a \leq 3$$

$$h(a) = \frac{2\sqrt{3}}{9} - 2, \quad h(3) = -2$$

hence $h(a) < 0$ for any $0 \leq a \leq 3$ and this concludes the proof.