PROBLEMS AND SOLUTIONS

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Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before April 30, 2009. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11397. *Proposed by Grahame Bennett, Indiana University, Bloomington, IN.* Let a, b, c, x, y, z be positive numbers such that $a+b+c = x+y+z$ and $abc = xyz$. Show that if $\max\{x, y, z\} \ge \max\{a, b, c\}$, then $\min\{x, y, z\} \ge \min\{a, b, c\}$.

11398. *Proposed by Stanley Huang, Jiangzhen Middle School, Huaining, China.* Suppose that acute triangle ABC has its middle-sized angle at A. Suppose further that the incenter I is equidistant from the circumcenter O and the orthocenter H. Show that angle A has measure 60 degrees and that the circumradius of IBC is the same as that of ABC.

11399. *Proposed by Biaggi Ricceri, University of Catania, Catania, Italy.* Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with finite nonzero measure M, and let $p > 0$. Let f be a lower semicontinuous function on $\mathbb{R}$ with the property that f has no global minimum, but for each $\lambda > 0$, the function $t \mapsto f(t) + \lambda|t|^p$ does have a unique global minimum. Show that exactly one of the two following assertions holds:
(a) For every $u \in L^p(\Omega)$ that is not essentially constant,

$$Mf\left(\left(\frac{1}{M}\int_\Omega |u(x)|^p \, d\mu\right)^{1/p}\right) < \int_\Omega f(u(x))\, d\mu,$$

and $f(t) < f(s)$ whenever $t > 0$ and $-t \le s < t$.
(b) For every $u \in L^p(\Omega)$ that is not essentially constant,

$$Mf\left(-\left(\frac{1}{M}\int_\Omega |u(x)|^p \, d\mu\right)^{1/p}\right) < \int_\Omega f(u(x))\, d\mu,$$

and $f(-t) < f(s)$ whenever $t > 0$ and $-t < s \le t$.

11400. *Proposed by Paul Bracken, University of Texas-Pan American, Edinburg, TX.* Let ζ be the Riemann zeta function. Evaluate $\sum_{n=1}^{\infty} \zeta(2n)/(n(n+1))$ in closed form.

11401. *Proposed by Marius Cavachi, "Ovidius" University of Constanţa, Constanţa, Romania.* Let A be a nonsingular square matrix with integer entries. Suppose that for every positive integer k, there is a matrix X with integer entries such that $X^k = A$. Show that A must be the identity matrix.

11402. *Proposed by Catalin Barboianu, Infarom Publishing, Craiova, Romania.* Let $f\colon [0,1] \to [0,\infty)$ be a continuous function such that $f(0) = f(1) = 0$ and $f(x) > 0$ for $0 < x < 1$. Show that there exists a square with two vertices in the interval (0,1) on the x-axis and the other two vertices on the graph of f.

11403. *Proposed by Yaming Yu, University of California Irvine, Irvine, CA.* Let n be an integer greater than 1, and let f_n be the polynomial given by

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j).$$

Find the degree of f_n.

SOLUTIONS

A Telescoping Fibonacci Sum

11258 [2006, 939]. *Proposed by Manuel Kauers, Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria.* Let F_n denote the nth Fibonacci number, and let i denote $\sqrt{-1}$. Prove that

$$\sum_{k=0}^{\infty} \frac{F_{3^k} - 2F_{1+3^k}}{F_{3^k} + iF_{2\cdot 3^k}} = i + \frac{1}{2}\left(1 - \sqrt{5}\right).$$

Solution by Richard Stong, Rice University, Houston, TX. Let ϕ denote the golden ratio $(1+\sqrt{5})/2$, and recall the Binet formula for the Fibonacci numbers:

$$F_n = \left(\phi^n + (-1)^{n-1}\phi^{-n}\right)/\sqrt{5}.$$

For odd m,

$$\sqrt{5}(F_m + iF_{2m}) = i\phi^{2m} + \phi^m + \phi^{-m} - i\phi^{-2m} = -i\phi^{-2m}(i\phi^m + 1)(i\phi^{3m} + 1)$$

and

$$F_m - 2F_{1+m} = \frac{1-2\phi}{\sqrt{5}}\phi^m + \frac{1+2\phi^{-1}}{\sqrt{5}}\phi^{-m} = -\phi^m + \phi^{-m}.$$

Hence

$$\frac{F_m - 2F_{1+m}}{F_m + iF_{2m}} = \sqrt{5}\frac{-i\phi^{3m} + i\phi^m}{(i\phi^m + 1)(i\phi^{3m} + 1)} = \left(\frac{\sqrt{5}}{i\phi^{3m} + 1} - \frac{\sqrt{5}}{i\phi^m + 1}\right).$$

Thus the desired sum telescopes as

$$\sum_{k=0}^{\infty} \frac{F_{3^k} - 2F_{1+3^k}}{F_{3^k} + iF_{2\cdot 3^k}} = \sum_{k=0}^{\infty}\left(\frac{\sqrt{5}}{i\phi^{3^{k+1}} + 1} - \frac{\sqrt{5}}{i\phi^{3^k} + 1}\right) = -\frac{\sqrt{5}}{i\phi + 1} = i - \phi^{-1}.$$

Also solved by S. Amghibech (Canada), M. R. Avidon, M. Bataille (France), R. Chapman (U. K.), C. K. Cook, M. Goldenberg & M. Kaplan, C. C. Heckman, G. C. Greubel, D. E. Iannucci, H. Kwong, O. P. Lossers (Netherlands), K. McInturff, C. R. Pranesachar (India), H. Roelants (Belgium), H.-J. Seiffert (Germany), A. Stadler (Switzerland), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

3-D Rotations and Translations

11276 [2007, 165]. *Proposed by Eugene Herman, Grinnell College, Grinnell, IA.* Let $T_1, \dots, T_n$ be translations in $\mathbb{R}^3$ with translation vectors $\mathbf{t}_1, \dots, \mathbf{t}_n$, and let R be a rotational linear transformation on $\mathbb{R}^3$ that rotates space through an angle of π/n about an axis parallel to a vector $\mathbf{r}$. Define a transformation C by $C = (RT_n \cdots RT_2RT_1)^2$. Prove that C is a translation, find an explicit formula for its translation vector in terms of $\mathbf{r}$, n, and $\mathbf{t}_1, \dots, \mathbf{t}_n$, and prove that there is a line ℓ in $\mathbb{R}^3$, independent of $\mathbf{t}_1, \dots, \mathbf{t}_n$, such that C translates space parallel to ℓ.

Solution by Mark D. Meyerson, US Naval Academy, Annapolis, MD. If $T_{\mathbf{v}}$ denotes translation by vector $\mathbf{v}$, then $RT_{\mathbf{v}} = T_{R\mathbf{v}}R$. Now R is a rotation through π/n, so R^n is a half turn about the axis of R and R^{2n} is the identity. If $\mathbf{v}$ is orthogonal to $\mathbf{r}$, then $R^n\mathbf{v} = -\mathbf{v}$ and $T_{\mathbf{v}+R^n\mathbf{v}}$ is the identity.

Let $\mathbf{p}_k$ be the vector projection of $\mathbf{t}_k$ onto $\mathbf{r}$, and let $\mathbf{o}_k$ be the orthogonal component, so that $\mathbf{t}_k = \mathbf{p}_k + \mathbf{o}_k$. Now $T_k = T_{\mathbf{p}_k}T_{\mathbf{o}_k}$. Furthermore, R takes $\mathbf{p}_k$ to itself, R and $T_{\mathbf{p}_k}$ commute, and any two translations commute.

Let $\mathbf{p} = 2(\mathbf{p}_n + \cdots + \mathbf{p}_2 + \mathbf{p}_1)$. We compute

$$C = (RT_n \cdots RT_2RT_1)^2 = (RT_{\mathbf{o}_n} \cdots RT_{\mathbf{o}_2}RT_{\mathbf{o}_1})^2(T_{\mathbf{p}_n} \cdots T_{\mathbf{p}_2}T_{\mathbf{p}_1})^2.$$

The second factor reduces to $T_{\mathbf{p}}$. In the first factor, move the rotations to the right:

$$\begin{aligned}
&(RT_{o_n} \cdots RT_{o_2}RT_{o_1})^2\\
&\quad= T_{R\mathbf{o}_n}RT_{R\mathbf{o}_{n-1}}R \cdots T_{R\mathbf{o}_2}RT_{R\mathbf{o}_1}RT_{R\mathbf{o}_n}RT_{R\mathbf{o}_{n-1}}R \cdots T_{R\mathbf{o}_2}RT_{R\mathbf{o}_1}R\\
&\quad= T_{R\mathbf{o}_n}T_{R^2\mathbf{o}_{n-1}}R \cdots RT_{R^2\mathbf{o}_2}RT_{R^2\mathbf{o}_1}RT_{R^2\mathbf{o}_n}RT_{R^2\mathbf{o}_{n-1}}R \cdots RT_{R^2\mathbf{o}_2}RT_{R^2\mathbf{o}_1}R^2\\
&\quad= \cdots = T_{R\mathbf{o}_n+R^{n+1}\mathbf{o}_n}T_{R^2\mathbf{o}_{n-1}+R^{n+2}\mathbf{o}_{n-1}} \cdots T_{R^{n-1}\mathbf{o}_2+R^{2n-1}\mathbf{o}_2}T_{R^n\mathbf{o}_1+R^{2n}\mathbf{o}_1}R^{2n};
\end{aligned}$$

the last expression is the identity. Hence $C = T_{\mathbf{p}}$, translation through twice the sum of the projections of the $\mathbf{t}_k$ onto $\mathbf{r}$, and ℓ can be any line parallel to $\mathbf{r}$.

Also solved by R. Bagby, M. Bataille (France), D. R. Bridges, R. Chapman (U. K.), K. Claassen, K. Dale (Norway), M. Englefield (Australia), A. Fok (Hong Kong), J.-P. Grivaux (France), J. A. Grzesik, G. Janusz, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Stong, T. Tam, E. I. Verriest, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Hofstra University Problem Solvers, and the proposer.

A Sinh of a Series

11286 [2007, 78]. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* Show that

$$\sum_{n=0}^{\infty} \left(e^{\pi/4} - (-1)^n \sinh(n + 1/2)\pi\right) e^{-n(n+1)\pi} = 0.$$

Solution by O. P. Lossers, Eindhoven University of Technology, The Netherlands. The C^∞ function

$$F(x, y) = \sum_{k=-\infty}^{\infty} e^{-(k+x)^2 y}, \qquad y > 0, \tag{1}$$

is even and periodic in x with period 1. Therefore it is the sum of a cosine series

$$F(x, y) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\pi x),$$

with $A_n = 2\int_{-\infty}^{\infty} e^{-x^2 y} \cos(2\pi nx)\,dx$. It follows that

$$F(x, y) = \sqrt{\frac{\pi}{y}} \left(1 + 2\sum_{n=1}^{\infty} e^{-\pi^2 n^2/y} \cos(2\pi nx)\right). \tag{2}$$

Substituting $x = 1/2$ and $y = \pi$ into the expression for $F(x, y)$ given in (1) leads to

$$F(1/2, \pi) = \sum_{k=-\infty}^{\infty} e^{-(k+1/2)^2\pi} = \sum_{k=-\infty}^{-1} e^{-(k+1/2)^2\pi} + e^{-\pi/4} + \sum_{k=1}^{\infty} e^{-(k+1/2)^2\pi}$$

$$= \sum_{k=2}^{\infty} e^{-(k-1/2)^2\pi} + 2e^{-\pi/4} + \sum_{k=1}^{\infty} e^{-(k+1/2)^2\pi} = 2\sum_{k=0}^{\infty} e^{-(k+1/2)^2\pi}.$$

Substituting into (2), we obtain

$$F(1/2, \pi) = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2} = \sum_{n=0}^{\infty}(-1)^n e^{-\pi n^2} + \sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2}$$

$$= \sum_{n=0}^{\infty}(-1)^n e^{-\pi n^2} + \sum_{n=0}^{\infty}(-1)^{n+1} e^{-\pi(n+1)^2}.$$

Equating these two expressions for F, we find that

$$2\sum_{k=0}^{\infty} e^{-(k+1/2)^2\pi} = \sum_{n=0}^{\infty}(-1)^n e^{-\pi n^2} + \sum_{n=0}^{\infty}(-1)^{n+1} e^{-\pi(n+1)^2}.$$

Multiplication of both sides by $e^{\pi/2}$ leads to

$$2\sum_{k=0}^{\infty} e^{\pi/4} e^{-k(k+1)\pi} = \sum_{n=0}^{\infty}(-1)^n e^{-\pi n(n+1)} \left(e^{n\pi+\pi/2} - e^{-n\pi-\pi/2}\right).$$

Applying the definition of the hyperbolic sine, we obtain the desired equation.

Also solved by R. Chapman (U.K.), J. Grivaux (France), O. Kouba (Syria), G. Lamb, M. A. Prasad (India), O. G. Ruehr, A. Stadler (Switzerland), R. Stong, J. Sun, FAU Problem Solving Group, GCHQ Problem Solving Group, and the proposer.

A Variant Intermediate Value

11290 [2007, 359]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal Highschool, Constanza, Romania.* Let f and g be continuous real-valued functions on $[0, 1]$. Prove that there exists c in $(0, 1)$ such that

$$\int_{x=0}^{1} f(x)\,dx \int_{x=0}^{c} xg(x)\,dx = \int_{x=0}^{1} g(x)\,dx \int_{x=0}^{c} xf(x)\,dx.$$

Solution by Kenneth F. Andersen, University of Alberta, Edmonton, AB, Canada. Observe first that if $h(x)$ is continouous on $[0, 1]$ and $H(x) = \int_0^x yh(y)\,dy$, then $H(x)$ is continuous on $[0, 1]$ with $\lim_{x\to 0^+} H(x)/x = 0$, so an integration by parts yields

$$\int_0^1 h(x)\,dx = \int_0^1 \frac{xh(x)}{x}dx = \frac{H(x)}{x}\bigg|_0^1 + \int_0^1 \frac{H(x)\,dx}{x^2}$$

$$= H(1) + \int_0^1 \frac{H(x)\,dx}{x^2} = \lim_{x\to 1^-} H(x) + \int_0^1 \frac{H(x)\,dx}{x^2}. \qquad (1)$$

Now suppose in addition that $\int_0^1 h(x) = 0$. By (1), $H(x)$ cannot be positive for all x in $(0, 1)$, nor can it be negative for all x in $(0, 1)$. Thus by the Intermediate Value Theorem there is a $c_h \in (0, 1)$ such that $H(c_h) = 0$. Now the required result may be deduced: if $\int_0^1 f(x)\,dx = 0$, then the result holds with $c = c_f$; if $\int_0^1 g(x)\,dx = 0$, then the result holds with $c = c_g$. Otherwise the result holds with $c = c_h$, where

$$h(x) = \frac{f(x)}{\int_0^1 f(y)\,dy} - \frac{g(x)}{\int_0^1 g(y)\,dy}.$$

Editorial comment. (i) The functions f and g need not be continuous—it is sufficient that they be integrable. This was observed by Botsko, Pinelis, Schilling, and Schmuland. (ii) Keselman, Martin, and Pinelis noted that $\int_0^1 xf(x)\,dx$ and $\int_0^1 xg(x)\,dx$ can be replaced with $\int_0^1 \phi(x)f(x)\,dx$ and $\int_0^1 \phi(x)g(x)\,dx$, where $\phi(x)$ satisfies suitable conditions—roughly speaking, that ϕ is differentiable and strictly monotonic, although the specific conditions vary from one of these solvers to another.

Also solved by U. Abel (Germany), S. Amghibech (Canada), M. W. Botsko & L. Mismas, R. Chapman (U. K.), J. G. Conlon & W. C. Troy, P. P. Dályay (Hungary), J. W. Hagood, E. A. Herman, S. J. Herschkorn, E. J. Ionascu, G. L. Isaacs, G. Keselman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), G. Martin (Canada), J. Metzger & T. Richards, M. D. Meyerson, A. B. Mingarelli & J. M. Pacheco & A. Plaza (Spain), E. Mouroukos (Greece), P. Perfetti (Italy), I. Pinelis, M. A. Prasad (India), K. Schilling, B. Schmuland (Canada), H.-J. Seiffert (Germany), J. Sun, R. Tauraso (Italy), M. Tetiva (Romania), L. Zhou, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

Double Integral

11295 [2007, 452]. *Proposed by Stefano Siboni, University of Trento, Trento, Italy.* For positive real numbers ϵ and ω, let M be the mapping of $[0, 1) \times [0, 1)$ into itself defined by $M(x, y) = (\{2x\}, \{y + \omega + \epsilon x\})$, where $\{u\}$ denotes $u - \lfloor u \rfloor$, the fractional part of u. For integers a and b, let $e_{a,b}(x, y) = e^{2\pi i(ax+by)}$. Let

$$C_n(a, b; p, q) = \int_{y=0}^1 \int_{x=0}^1 e_{a,b}(M^n(x, y))\overline{e_{p,q}(x, y)}\,dx\,dy.$$

Show that $C_n(a, b; p, q) = 0$ if $q \neq b$, while $C_n(a, b; p, b)$ is given by

$$(-1)^a e_{b,b}(\omega n, \epsilon n/2)\frac{\sin\left[\pi(a + \epsilon b - 2^{-n}(p + \epsilon b))\right]}{\pi\,(a + \epsilon b - 2^{-n}(p + \epsilon b))}\prod_{j=0}^n \cos\left[\pi(\epsilon b - 2^{-j}(p + \epsilon b)\right].$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. Let $e(z) = e^{2\pi i z}$. For $n \geq 1$ we have

$$M^n(x, y) = \left(\left\{2^n x\right\}, \left\{y + n\omega + \epsilon(\{x\} + \{2x\} + \cdots + \{2^{n-1}x\})\right\}\right).$$

Since a and b are integers, the outer braces do not matter for $C_n(a, b; p, q) =$

$$\int_0^1 \int_0^1 e\big((2^n a - p)x + (b-q)y + bn\omega + b\epsilon(\{x\} + \{2x\} + \cdots + \{2^{n-1}x\})\big)\, dx\, dy.$$

Interchanging the order of integration shows that $C_n(a, b; p, q) = 0$ if $q \neq b$ and that

$$C_n(a, b; p, b) = e(b\omega n) \int_0^1 e\big((2^n a - p)x + b\epsilon(\{x\} + \{2x\} + \cdots + \{2^{n-1}x\})\big)\, dx.$$

For $n \geq 1$ define

$$F_n(\alpha, \beta) = \int_0^1 \exp\big(i\alpha x + 2i\beta x + i\beta(\{2x\} + \cdots + \{2^{n-1}x\})\big)\, dx,$$

so that $C_n(a, b; p, b) = e(b\omega n) F_n(2\pi(2^n a - p - \epsilon b), 2\pi\epsilon b)$. Now

$$F_1(\alpha, \beta) = \frac{e^{i(\alpha+2\beta)} - 1}{i(\alpha + 2\beta)} = \frac{e^{i(\beta+\alpha/2)}}{\beta/2 + \alpha/4} \cos\left(\frac{\beta}{2} + \frac{\alpha}{4}\right) \sin\left(\frac{\beta}{2} + \frac{\alpha}{4}\right).$$

For $n > 1$, the last term in the exponent has period $1/2$, so we split the integral into two parts to obtain

$$F_n(\alpha, \beta) = \big(e^{i(\alpha/2+\beta)} + 1\big) \int_0^{1/2} \exp\big(i(\alpha + 4\beta)x + i\beta(\{4x\} + \cdots + \{2^{n-1}x\})\big)\, dx,$$

which gives the recurrence

$$F_n(\alpha, \beta) = e^{i(\beta/2+\alpha/4)} \cos\left(\frac{\beta}{2} + \frac{\alpha}{4}\right) F_{n-1}\left(\frac{\alpha}{2}, \beta\right).$$

Repeated use of this, together with the formula for F_1, leads to

$$F_n(\alpha, \beta) = \frac{e^{i\beta(n+1/2)+i\alpha/2}}{\beta/2 + \alpha/2^{n+1}} \sin\left(\frac{\beta}{2} + \frac{\alpha}{2^{n+1}}\right) \prod_{j=1}^{n} \cos\left(\frac{\beta}{2} + \frac{\alpha}{2^{j+1}}\right).$$

Finally, $C_n(a, b; p, b) = e^{2\pi i b\omega n} F_n(2\pi(2^n a - p - \epsilon b), 2\pi\epsilon b)$ yields the required result after some simplifications based on the assumption that a and b are integers.

Also solved by R. Chapman (U. K.), D. Fleischman, GCHQ Problem Solving Group, and the proposer.

A Tricky Minimum

11297 [2007, 452]. *Proposed by Marian Tetiva, Bîrlad, Romania.* For positive a, b, and c, let

$$E(a, b, c) = \frac{a^2b^2c^2 - 64}{(a+1)(b+1)(c+1) - 27}.$$

Find the minimum value of $E(a, b, c)$ on the set D consisting of all positive triples (a, b, c) such that $abc = a + b + c + 2$, other than $(2, 2, 2)$.

Solution by John H. Lindsey II, Cambridge, MA. Let m be the geometric mean, defined by $m = (abc)^{1/3}$. By the arithmetic-geometric mean inequality, $a + b + c \geq 3m$, with equality if and only if $a = b = c$. Thus $m^3 = a + b + c + 2 \geq 3m + 2$, or $(m - 2)(m + 1)^2 = m^3 - 3m - 2 \geq 0$ and hence $m \geq 2$. Equality forces $a = b = c = 2$, so in fact $m > 2$. Using the arithmetic-geometric mean inequality again to obtain $ab + bc + ca \geq 3((ab)(bc)(ca))^{1/3} = 3m^2$, we see that $(a + 1)(b + 1)(c + 1) > m^3 + 3m^2 + 3m + 1 > 27$. Thus the numerator and denominator of E are always positive on D.

For fixed $(a, b, c) \in D$, consider all triples $(a', b', c') \in D$ with $a'b'c' = abc$. For such triples the numerator of E is fixed and the denominator will be maximized (hence E minimized) if we maximize $a'b' + b'c' + c'a'$. Since $a' + b' + c' = a + b + c$ is fixed (at $abc - 2$), a', b', and c' are bounded above; since also $a'b'c'$ is fixed and positive, they are bounded away from zero. Thus they form a closed bounded set. Hence we may choose a', b', c' to maximize $a'b' + b'c' + c'a'$.

Suppose this maximum occurs for a', b', c' distinct. By symmetry we may assume that $a' < b' < c'$. Let $f(x) = (x - a')(x - b')(x - c')$. When ϵ is positive and sufficiently small, $f(x) + \epsilon x$ has three distinct positive roots with the same sum and product as a, b, c (since f is a cubic polynomial), but this contradicts maximality of the denominator. Thus two of a', b', c' must be equal. Hence it suffices to minimize E under the additional constraint $a = b$. In this case the condition $abc = a + b + c + 2$ gives $c = \frac{2}{a-1}$ and we compute

$$E\left(a, a, \frac{2}{a-1}\right) = \frac{4(a^2 + 4a - 4)}{(a + 7)(a - 1)} = 4 - \frac{17}{2(a + 7)} + \frac{1}{2(a - 1)}$$

and

$$\frac{d}{da}E\left(a, a, \frac{2}{a-1}\right) = \frac{17}{2(a + 7)^2} - \frac{1}{2(a - 1)^2}.$$

The unique critical point occurs at $a = \frac{3+\sqrt{17}}{2}$ where $E = \frac{23+\sqrt{17}}{8} \approx 3.390388$. As $a \to 1^+$, $E \to \infty$ and as $a \to \infty$, $E \to 4$ so this is the minimum of E.

Also solved by A. Alt, J. Grivaux (France), E. A. Herman, G. I. Isaacs, K.-W. Lau (China), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

Errata and End Notes for 2008.

An Infinite Product Based on a Base

11222 [2006, 459]. *Proposed by Jonathan Sondow, New York, NY.* Fix an integer $B \geq 2$, and let $s(n)$ denote the sum of the base-B digits of n. Prove that

$$\prod_{n=0}^{\infty} \prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{nB + k}{nB + k + 1}\right)^{(-1)^{s(n)}} = \frac{1}{\sqrt{B}}.$$

Solution by the proposer. Set $\epsilon(n) = (-1)^{s(n)}$. If B is odd, then $s(n) \equiv n \bmod 2$, since all powers of B are odd. If B is even, then the constant term in the base-B expansion of $2m$ cannot be $B - 1$, and hence $s(2m + 1) = s(2m) + 1$. Hence in both cases $\epsilon(2m + 1) = -\epsilon(2m)$. Let $\delta_k = 1$ if $k = 0$, and otherwise $\delta_k = 0$, and let

$$P_{k,B} = P_k = \prod_{n=\delta_k}^{\infty} \left(\frac{nB + k}{nB + k + 1}\right)^{\epsilon(n)}.$$

Then P_k converges because it is a product over $m \geq 0$ of factors of the form

$$\Big(\frac{(2m)B+k+1}{(2m)B+k} \cdot \frac{(2m+1)B+k}{(2m+1)B+k+1}\Big)^{\pm 1},$$

which simplifies to $(1 + \frac{B+1}{x(x+B+1)})^{\pm 1}$, where $x = 2mB + k$. Thus products of the form $\prod_{k \in S} P_k$ converge for any finite subset S of $[0, B-1]$, and in particular, the original product converges.

We now consider a product of P_k's that telescopes nicely:

$$\begin{aligned} \prod_{k=0}^{B-1} \prod_{n=\delta_k}^{\infty} \Big(\frac{nB+k}{nB+k+1}\Big)^{\epsilon(n)} &= \prod_{k=1}^{B-1} \Big(\frac{k}{k+1}\Big)^{\epsilon(0)} \prod_{n=1}^{\infty} \prod_{k=0}^{B-1} \Big(\frac{nB+k}{nB+k+1}\Big)^{\epsilon(n)} \\ &= \frac{1}{B} \prod_{m=1}^{\infty} \Big(\frac{m}{m+1}\Big)^{\epsilon(m)}. \end{aligned} \tag{1}$$

If $0 \leq k < B$, then $s(nB+k) = s(n) + k$, so $\epsilon(nB+k) = (-1)^k \epsilon(n)$. After splitting the last product in (1) by collecting factors with the same residue modulo B, apply $\epsilon(nB+k) = (-1)^k \epsilon(n)$ to obtain

$$\prod_{m=1}^{\infty} \Big(\frac{m}{m+1}\Big)^{\epsilon(m)} = \prod_{k=0}^{B-1} \prod_{n=\delta_k}^{\infty} \Big(\frac{nB+k}{nB+k+1}\Big)^{(-1)^k \epsilon(n)}.$$

Substitute this into (1). Since the infinite products are all nonzero (being convergent and having no zero factors), the factors for even k are the same on both sides and cancel out. This yields

$$\prod_{\substack{k \text{ odd} \\ 0<k<B}} \prod_{n=0}^{\infty} \Big(\frac{nB+k}{nB+k+1}\Big)^{\epsilon(n)} = \frac{1}{B} \prod_{\substack{k \text{ odd} \\ 0<k<B}} \prod_{n=0}^{\infty} \Big(\frac{nB+k}{nB+k+1}\Big)^{-\epsilon(n)}.$$

All the products are positive, and the desired formula follows.

Editorial comment. The solution of Problem 11222 for odd values of B given in the May, 2008 issue of the *Monthly* was selected at a time when the proposer's solution had become separated from the file of solutions. The previously published solution treated odd B only by reference to the literature. The proposer's elegant solution covers all cases simultaneously and efficiently. Fortunately, it was recovered, and we are pleased to present it.

The names of solver Apostolis Demis, of Athens, Greece (**11285**, [2007,358]), and William Dickinson, (**11201**, [2008,73]) were misspelled. Our apologies.

Paolo Perfetti gives a counterexample to the if-direction of part (a) in **11257**, [2008, 269]. If $z_k = (-1)^{k-1}/\ln(k+1)$ (and $s_k = \sum_1^j z_k$), then $\langle s_n \rangle$ converges, but $\sum z_k/s_k$ diverges, tending to $-\infty$.