Junior problems

J139. Let $a_0 = a_1 = 1$ and

$$a_{n+1} = \frac{a_n^2}{a_n + a_{n-1}}$$

for $n \geq 1$. Find a_n in closed form.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Dmitri Skjorshammer, Harvey Mudd College, USA

We prove that $a_n = \frac{1}{n!}$ by induction on n.
Base Case: Consider $n = 1$. Then $a_2 = \frac{a_1^2}{a_1+a_0} = \frac{1}{2} = \frac{1}{2!}$, as desired.
Strong Induction Hypothesis: Suppose that for $n \leq k$, it is true that $a_k = \frac{1}{k!}$.
Inductive Step: Consider $k+1$. Then

$$\begin{aligned} a_{k+1} = \frac{a_k^2}{a_k + a_{k-1}} &= \frac{\left(\frac{1}{k!}\right)^2}{\frac{1}{k!} + \frac{1}{(k-1)!}} \\ &= \frac{\left(\frac{1}{k!}\right)^2}{\frac{1+k}{k!}} = \left(\frac{1}{k!}\right)^2 \frac{k!}{k+1} \\ &= \frac{1}{k!(k+1)} = \frac{1}{(k+1)!}. \end{aligned}$$

Since the claim is true for the base case and for the inductive step, it follows that it is true for all $n \geq 1$.

Also solved by G. C. Greubel, Newport News, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Ercole Suppa, Teramo, Italy; Arkady Alt, San Jose, California, USA; Badr Alghamdi, Saudi Arabia; Bedri Hajrizi, Albania; Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Tarik Adnan Moon, Bangladesh.

J140. Let n be a positive integer. Find all real numbers x such that

$$\lfloor x\rfloor + \lfloor 2x\rfloor + \ldots + \lfloor nx\rfloor = \frac{n(n+1)}{2}.$$

Proposed by Mihai Piticari, "Dragos-Voda" National College, Romania

First solution by John Mangual

Let $f(x) = \lfloor x\rfloor + \lfloor 2x\rfloor + \cdots + \lfloor nx\rfloor$. Note $f(1) = n(n+1)/2$ and

$$f(1-\epsilon) = \lfloor (1-\epsilon)\rfloor + \lfloor 2(1-\epsilon)\rfloor + \cdots + \lfloor n(1-\epsilon)\rfloor < 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for $\epsilon > 0$. Since $f(x)$ is monotone increasing, $x = 1$ is the smallest solution to $f(x) = n(n+1)/2$. $f(x)$ will continue to hold this value until one of the $\lfloor kx\rfloor$ increases at $x = 1 + 1/k$. The first time this happens is at $x = 1 + 1/n$. Therefore the solution set is the interval $[1, 1/n)$.

Second solution by Michel Bataille, France

Let $S(x) = \lfloor x\rfloor + \lfloor 2x\rfloor + \cdots + \lfloor nx\rfloor$. We show that $S(x) = \frac{n(n+1)}{2}$ if and only if $1 \leq x < 1 + \frac{1}{n}$.

First, suppose $1 \leq x < 1 + \frac{1}{n}$ and let $k \in \{1, 2, \ldots, n\}$. Then

$$k \leq kx < k + \frac{k}{n} \leq k + 1$$

so that $\lfloor kx\rfloor = k$. It follows that $S(x) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
If $x < 1$, then $kx < k$ so that $\lfloor kx\rfloor < k$ for every $k \in \{1, 2, \ldots, n\}$ and by addition, $S(x) < \frac{n(n+1)}{2}$. Thus, no $x < 1$ is a solution.
Lastly, suppose that $x \geq 1 + \frac{1}{n}$. Then, for every $k \in \{1, 2, \ldots, n-1\}$, we have $kx \geq k + \frac{k}{n}$ so that $\lfloor kx\rfloor \geq k$. Besides, $nx \geq n + 1$, hence $\lfloor nx\rfloor \geq n + 1$. It follows that

$$S(x) \geq 1 + 2 + \cdots (n-1) + (n+1) = \frac{n(n+1)}{2} + 1,$$

which implies that such an x is not a solution.

Also solved by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Ercole Suppa, Teramo, Italy; Arkady Alt, San Jose, California, USA; Bedri Hajrizi, Albania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Tarik Adnan Moon, Bangladesh.

J141. Let a, b, c be the side lengths of a triangle. Prove that

$$0 \le \frac{a-b}{b+c} + \frac{b-c}{c+a} + \frac{c-a}{a+b} < 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, "Babes-Bolyai" University, Romania

First solution by the authors. We can write

$$\sum_{cyc} \frac{a-b}{b+c} = \sum_{cyc} \frac{a+c}{b+c} - 3 = E - 3,$$

where

$$E = \frac{a+c}{b+c} + \frac{b+a}{b+c} + \frac{c+b}{b+c}.$$

For the right-hand side inequality observe that in any triangle we have $b + c > \frac{1}{2}(a+b+c)$, $c + a > \frac{1}{2}(a+b+c)$, and $a + b > \frac{1}{2}(a+b+c)$. It follows

$$E < \frac{2(a+c+b+a+c+b)}{a+b+c} = 4.$$

For the left-hand side inequality we use the Cauchy-Schwarz inequality and get

$$E = \sum_{cyc} \frac{a+c}{b+c} = \sum_{cyc} \frac{(a+c)^2}{(a+c)(b+c)} \ge \frac{(\sum_{cyc}(a+c))^2}{\sum_{cyc}(a+c)(b+c)} = \frac{4(a+b+c)^2}{a^2+b^2+c^2+3(ab+bc+ca)}.$$

The last fraction is greater than 3, since we have $a^2 + b^2 + c^2 \ge ab + bc + ca$. The equality holds if and only if the triangle is equilateral.

Second solution by Arkady Alt, San Jose, California, USA

Since

$$\begin{aligned}
\sum_{cyc} (a-b)(a+b)(c+a) &= \sum_{cyc} (a-b)\left(a^2+ab+bc+ca\right) \\
&= \sum_{cyc} (a-b)\,a^2 + (ab+bc+ca)\sum_{cyc}(a-b) \\
&= \sum_{cyc} (a-b)\,a^2 = a^3+b^3+c^3-a^2b-b^2c-c^2a \\
&= \frac{1}{3}\sum \left(2a^3+b^3-3a^2b\right) = \frac{1}{3}\sum (a-b)^2(2a+b)
\end{aligned}$$

then

$$\sum_{cyc} \frac{a-b}{b+c} = \frac{a^3+b^3+c^3-a^2b-b^2c-c^2a}{(a+b)(b+c)(c+a)} \geq 0.$$

It remains to prove that

$$\sum_{cyc} \frac{a-b}{b+c} < 1 \iff a^3+b^3+c^3-a^2b-b^2c-c^2a < (a+b)(b+c)(c+a).$$

We have

$$\begin{aligned}&(a+b)(b+c)(c+a)-a^3-b^3-c^3+a^2b+b^2c+c^2a\\&=2abc+a^2b+b^2c+c^2a+\sum_{cyc} a^2(b+c-a)) > 0\end{aligned}$$

because a, b, c satisfy the inequalities $b+c-a > 0, c+a-b > 0, a+b-c > 0$.

Third solution by Michel Bataille, France

The central expression rewrites as $\frac{N}{D}$ with

$$N = (a-b)a^2 + (b-c)b^2 + (c-a)c^2 \quad \text{and} \quad D = (a+b)(b+c)(c+a).$$

Assume without loss of generality that $a = \max\{a, b, c\}$. Then,

$$N = (a-b)a^2+(b-c)b^2+(c-b)c^2+(b-a)c^2 = (a-b)(a-c)(a+c)+(b-c)^2(b+c) \geq 0$$

and since $D > 0$, we obtain $\frac{N}{D} \geq 0$.
It is easily checked that

$$D - N = a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) + a^2b + b^2c + c^2a + 2abc,$$

hence $D - N > 0$ (since a, b, c are the side lengths of a triangle, we have $a < b+c, b < c+a$ and $c < a+b$). Thus, $\frac{N}{D} < 1$ and we conclude that $0 \leq \frac{N}{D} < 1$, as required.

Also solved by G. C. Greubel, Newport News, USA; Ercole Suppa, Teramo, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

J142. For each positive integer m, define the binomial coefficient $\binom{x}{m} = \frac{x(x-1)\cdots(x-m+1)}{m!}$. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_1 + x_2 + \cdots + x_n \geq n^2$. Prove that

$$\frac{n-1}{2}\left(\sum_{i=1}^{n}\binom{x_i}{3}\right)\left(\sum_{i=1}^{n}x_i\right) \geq \frac{n-2}{3}\left(\sum_{i=1}^{n}\binom{x_i}{2}\right)^2.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by G. C. Greubel, Newport News, USA

Let

$$\begin{aligned}\phi &= \frac{n-1}{2}\left(\sum_{i=1}^{n}\binom{x_i}{3}\right)\left(\sum_{i=1}^{n}x_i\right) &(1)\\ &= \frac{n-1}{2}\left(\sum_{i=1}^{n}\frac{x_i(x_i-1)(x_i-2)}{3!}\right)\left(\sum_{i=1}^{n}x_i\right)\\ &= \frac{n-1}{12}\left(\sum_{i=1}^{n}(x_i^3 - 3x_i^2 + 2x_i)\right)\left(\sum_{i=1}^{n}x_i\right). &(2)\end{aligned}$$

When use is made of Chebychev's inequality of two and three variables, namely,

$$\sum_{k=1}^{n}x_k y_k \geq n\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}y_k\right) \qquad (3)$$

and

$$\sum_{k=1}^{n}x_k y_k z_k \geq n\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}y_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}z_k\right) \qquad (4)$$

we have

$$\sum_{k=1}^{n}x_i \geq n^2, \qquad (5)$$

by the original problem, and

$$\begin{aligned}\sum_{k=1}^{n}x_i^2 &\geq n\left(\frac{1}{n}(n^2)\right)^2 = n^3 &(6)\\ \sum_{k=1}^{n}x_i^3 &\geq n\left(\frac{1}{n}(n^2)\right)^3 = n^4. &(7)\end{aligned}$$

From this we have

$$\begin{aligned}\phi &\geq \frac{n-1}{12}\left(n^4-3n^3+2n^2\right)\left(n^2\right)\\ &\geq \frac{n^4}{12}(n-1)^2(n-2).\end{aligned} \tag{8}$$

This can be rearranged to be in the form

$$\begin{aligned}\phi &\geq \frac{n-2}{3}\cdot\frac{n^4(n-1)^2}{4}\\ &\geq \frac{n-2}{3}\left(\frac{n^2}{2}(n-1)\right)^2\\ &\geq \frac{n-2}{3}\left(\sum_{k=1}^{n}\frac{x_i(x_i-1)}{2!}\right)^2\\ &\geq \frac{n-2}{3}\left(\sum_{k=1} n\binom{x_i}{2}\right)^2.\end{aligned} \tag{9}$$

Thus we have

$$\frac{n-1}{2}\left(\sum_{i=1}^{n}\binom{x_i}{3}\right)\left(\sum_{i=1}^{n}x_i\right) \geq \frac{n-2}{3}\left(\sum_{k=1} n\binom{x_i}{2}\right)^2. \tag{10}$$

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

If $n=1$, the RHS is negative while the LHS is zero, or we may assume that $n\geq 2$. Denote $S_k = x_1^k + x_2^k + \cdots + x_n^k$. Clearly,

$$\sum_{i=1}^{n}\binom{x_i}{3} = \frac{S_3-3S_2+2S_1}{6}, \qquad \sum_{i=1}^{n}\binom{x_i}{2} = \frac{S_2-S_1}{2},$$

or the proposed inequality rewrites as

$$S_1S_3 - S_2^2 - S_1S_2 + S_1^2 + \frac{(S_2-S_1)^2}{n-1} \geq 0.$$

Now, $S_2^2-(n+1)S_1S_2+nS_1^2 = (S_2-S_1)(S_2-nS_1)\geq 0$ as a consequence of the inequality between arithmetic and quadratic means, with equality iff all x_i are equal, because $S_2 \geq \frac{S_1^2}{n} \geq nS_1$. Thus, $(S_2-S_1)^2 \geq (n-1)(S_1S_2-S_1^2)$. Moreover,

$$S_1S_3 - S_2^2 = \sum_{i\neq j} x_ix_j(x_i-x_j)^2 \geq 0.$$

The conclusion follows, equality holding iff all x_i are equal.

J143. Let $x_1 = -2$, $x_2 = -1$ and

$$x_{n+1} = \sqrt[3]{n(x_n^2+1)+2x_{n-1}}$$

for $n \geq 2$. Find x_{2009}.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Badr Alghamdi, Saudi Arabia

By plugging in values we find

$$x_3 = 0, x_4 = 1, x_5 = 2.$$

We will prove that

$$x_n = n - 3$$

by induction on n. Assume it is true for n and let us prove it for *n+1*.

$$x_{n+1} = \sqrt[3]{n(x_n^2+1)+2x_{n-1}} = \sqrt[3]{n\left((n-3)^2+1\right)+2\,(n-4)}$$
$$= \sqrt[3]{n^3-6n^2+12n-8} = \sqrt[3]{(n-2)^3} = n-2$$

hence the inductive step is also true. We can now find $x_{2009} = 2009 - 3 = 2006$.

Also solved by G. C. Greubel, Newport News, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Ercole Suppa, Teramo, Italy; Arkady Alt, San Jose, California, USA; Bedri Hajrizi, Albania; Dmitri Skjorshammer, Harvey Mudd College, USA; Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

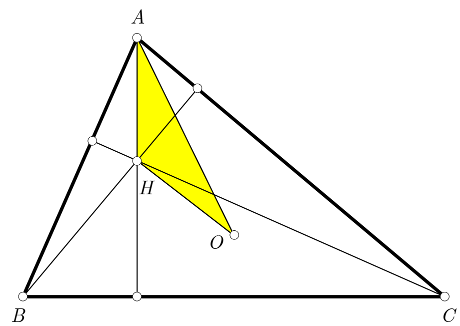
J144. Let ABC be a triangle with $a > b > c$. Denote by O and H its circumcenter and orthocenter, respectively. Prove that

$$\sin\angle AHO + \sin\angle BHO + \sin\angle CHO \leq \frac{(a-c)(a+c)^3}{4abc \cdot OH}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Ercole Suppa, Teramo, Italy

Let R, A, B, C, a, b, c be the circumradius, the angles and the side lengths of the triangle ABC, respectively.



Clearly we have

$$\angle HAO = \angle HAC - \angle OAC = (90^\circ - C) - (90^\circ - B) = B - C \tag{1}$$

The sine law in triangle AHO yields

$$\frac{AO}{\sin\angle AHO} = \frac{OH}{\sin\angle HAO} \tag{2}$$

From (1) and (2), taking into account the law of sines and the law of cosines in triangle ABC, it follows that

$$\begin{aligned}
\sin\angle AHO &= \frac{R}{OH}\cdot\sin(B-C) = \\
&= \frac{R}{OH}\cdot(\sin B\cos C - \cos B\sin C) = \\
&= \frac{1}{OH}\cdot\left(\frac{b}{2}\cos C - \frac{c}{2}\cos B\right) = \\
&= \frac{1}{OH}\cdot\left(b\cdot\frac{a^2+b^2-c^2}{4ab} - c\cdot\frac{a^2+c^2-b^2}{4ac}\right) = \\
&= \frac{1}{OH}\cdot\frac{b^2-c^2}{2a}
\end{aligned}$$

Building up two similar equalities and adding up all of them, we get

$$\begin{aligned}
&\sin\angle AHO + \sin\angle BHO + \sin\angle CHO = \\
&= \frac{1}{OH}\cdot\left(\frac{b^2-c^2}{2a} + \frac{a^2-c^2}{2b} + \frac{a^2-b^2}{2c}\right) = \\
&= \frac{1}{2abc\cdot OH}\left[bc\left(b^2-c^2\right) + ac\left(a^2-c^2\right) + ab\left(a^2-b^2\right)\right] = \\
&= \frac{1}{2abc\cdot OH}(a+b)(b+c)(a-c)(a-b+c) = \\
&= \frac{1}{4abc\cdot OH}(a+b)(b+c)(a-c)(2a-2b+2c)
\end{aligned}$$

Then, according to the above relation, the given inequality can be rewritten in the form

$$(a+b)(b+c)(2a-2b+2c) \le (a+c)^3$$

which is true because of **AM-GM** inequality. □

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We will show that the inequality holds always strictly. Applying the Sine Law to triangle ΔAHO, we find $OH\sin\angle AHO = AO\sin\angle OAH = R\sin(B-C)$, and similarly for its cyclic permutations, or the strict inequality is equivalent to

$$4abcR\left(\sin(A-B)+\sin(A-C)+\sin(B-C)\right) < (a-c)(a+c)^3.$$

Using the Cosine Law and trigonometric relations yields

$$(a-c)(a+c) = a^2-c^2 = b^2-2bc\cos A = b(a\cos C - c\cos A) = 2bR\sin(A-C),$$

while $\sin(A-B)+\sin(B-C) = 2\sin\frac{A-C}{2}\sin\frac{3B}{2}$, or the inequality is equivalent to

$$2ac\sin\frac{3B}{2} < \cos\frac{A-C}{2}(a^2+c^2).$$

Now, $2\sin\frac{3B}{2}\cos\frac{B}{2} = \sin(2B)+\sin B$, while $2\cos\frac{A-C}{2}\cos\frac{B}{2} = \sin A + \sin C$, or it suffices to prove that

$$(a+c)(a^2+c^2) > 2ac(b+2b\cos B), \qquad a^3+2b^3+c^3+a^2c+ac^2 > 2b(a^2+ac+c^2).$$

Write $2b = \rho(a+c)$, where $\rho < 2$ because of the triangular inequality. The proposed inequality transforms finally into

$$(\rho^3-3\rho+2)(a+c)^2 + (a-c)^2(2-\rho) > 0.$$

The second term is stricly positive, while $\rho^3-3\rho+2 = (\rho-1)^2(\rho+2) \ge 0$, with equality iff $\rho = 1$. The conclusion follows.

Senior problems

S139. Let $a_0 = 1$ and $a_{n+1} = a_0 \cdots a_n + 3$ for $n \geq 0$. Prove that

$$a_n + \sqrt[3]{1 - a_n a_{n+1}} = 1,$$

for all $n \geq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Observe that $a_{n+1} = (a_n - 3)a_n + 3 = a_n^2 - 3a_n + 3$ is an easy consequence of $a_{n+1} = a_0 \cdots a_n + 3$. It follows $a_n + \sqrt[3]{1 - a_n(a_n^2 - 3a_n + 3)} = 1$ that is $a_n + \sqrt[3]{1 - a_n)^3} = 1$ and we are done.

Also solved by G. C. Greubel, Newport News, USA; Arkady Alt, San Jose, California, USA; Dmitri Skjorshammer, Harvey Mudd College, USA; Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S140. Let a, b, c be integers. Prove that

$$\sum_{cyc}(a-b)(a^2+b^2-c^2)c^2$$

is divisible by $(a+b+c)^2$.

Proposed by Dorin Andrica, "Babeş-Bolyai" University, Cluj-Napoca, Romania

First solution by G. C. Greubel, Newport News, USA

Let ϕ be the cyclical sum of the problem, namely,

$$\begin{aligned}\phi &= \sum_{cycl}(a-b)(a^2+b^2-c^2)c^2 &(11)\\ &= (a-b)(a^2+b^2-c^2)c^2+(b-c)(b^2+c^2-a^2)a^2\\ &\qquad + (c-a)(c^2+a^2-b^2)b^2. &(12)\end{aligned}$$

Multiplying out the terms of (2) yields

$$\begin{aligned}\phi &= a^3(c^2-b^2)+b^3(a^2-c^2)+c^3(b^2-a^2)+a^4(c-b)+b^4(a-c)\\ &\quad + c^4(b-a)+abc[c(b-a)+a(c-b)+b(a-c)]\\ &= a^4(c-b)+a^3(c^2-b^2)+b^4(a-c)+b^3(a^2-c^2)\\ &\quad + c^4(b-a)+c^3(b^2-a^2)\\ &= (a+b+c)[a^3(c-b)+b^3(a-c)+c^3(b-a)]. &(13)\end{aligned}$$

Let

$$\sigma = a^3(c-b)+b^3(a-c)+c^3(b-a). \qquad (14)$$

Then

$$\begin{aligned}\sigma &= a^3(c-b)+b^3(a-c)+c^3(b-a)\\ &= ac(a^2-c^2)+ab(b^2-a^2)+bc(c^2-b^2)\\ &= (a+b+c)[ac(a-c)+ab(b-a)+bc(c-b)]\\ &\quad - abc[(a-c)+(b-a)+(c-b)]\\ &= (a+b+c)[ab(b-a)+bc(c-b)+ca(a-c)]\\ &= (a+b+c)\sum_{cycl} ab(b-a). &(15)\end{aligned}$$

From (1), (3), and (5) we have

$$\sum_{cycl}(a-b)(a^2+b^2-c^2)c^2 = (a+b+c)^2\sum_{cycl}ab(b-a). \tag{16}$$

This provides the fact that the original sum is divisible by $(a+b+c)^2$.

Second solution by Michel Bataille, France

We introduce the following polynomial with integer coefficients:

$$P(x) = x^4 + x^3(b+c) - x^2(b^2+c^2+bc) - x(b+c)(b^2+c^2) + bc(b+c)^2.$$

A simple calculation shows that

$$\sum_{cyc}(a-b)(a^2+b^2-c^2)c^2 = (c-b)P(a).$$

Now, assume that we have proved that $P(x)$ is divisible by $(x+b+c)^2$. The quotient $Q(x)$ has integer coefficients and the relation $P(a) = (a+b+c)^2Q(a)$ then implies that the given sum is divisible by $(a+b+c)^2$. Thus, it is sufficient to prove that $P(x)$ is divisible by $(x+b+c)^2$, or equivalently, that

$$P(-(b+c)) = P'(-(b+c)) = 0$$

where $P'(x) = 4x^3+3x^2(b+c)-2x(b^2+c^2+bc)-(b+c)(b^2+c^2)$ is the derivative of $P(x)$. Now, we compute

$$P(-(b+c)) = (b+c)^4-(b+c)^4-(b+c)^2(b^2+c^2+bc)+(b+c)^2(b^2+c^2)+bc(b+c)^2 = 0$$

and

$$P'(-(b+c)) = -4(b+c)^3+3(b+c)^3+2(b+c)(b^2+c^2+bc)-(b+c)(b^2+c^2) = 0$$

and the proof is complete.

Also solved by Arkady Alt, San Jose, California, USA; Ercole Suppa, Teramo, Italy; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S141. Four squares are laying inside a circle of radius $\sqrt{5}$ such that no two have a common point. Prove that one can place these squares inside a square of side 4, such that no two have a common point.

Proposed by Nairi Sedrakyan, Armenia

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote by $a \geq b \geq c \geq d$ the sidelengths of the four squares, by $ABCD$ the square with sidelength 4, and assume that $a+b<4$. Place the square with sidelength a with one vertex on A, and two sides along segments AB and AD, then displace it a distance $\delta = \frac{4-a-b}{3}$ towards the interior of $ABCD$ in both directions perpendicular to the sides of $ABCD$, and proceed similarly with the squares with sidelengths b, c, d on vertices B, C, D, respectively. Clearly, two squares will have common points iff $4 \leq a+b+\delta = \frac{4+2a+2b}{3}$, clearly not true since $a+b<4$. We can see that the squares may be placed inside a square of side 4 without common points, if any two squares have sidelengths adding up to less than 4.

Assume now that two squares $ABCD$ with sidelength a and $A'B'C'D'$ with sidelength b, such that $a+b \geq 4$, may be fitted (even touching each other) inside a circle with radius $\sqrt{5}$. Note that if they fit without touching, they may be brought into contact and still will fit inside the circle, or we may assume wlog that C is on side $C'D'$, where $CD' = d$ and $CC' = b-d$. Denote $\frac{\pi}{4} \leq \alpha = \angle ACC' \leq \frac{\pi}{2}$. We find

$$AA'^2 = b^2 + d^2 + 2a^2 + 2\sqrt{2}a(d\cos\alpha + b\sin\alpha),$$

$$BB'^2 = 2b^2 - 2bd + d^2 + a^2 + \sqrt{2}a(d\cos\alpha + (2b-d)\sin\alpha).$$

Given d, note that AA' is smallest when α is smallest because $d \leq b$, while BB' is smallest when α is closest to its upper bound $\frac{\pi}{2}$. Therefore, if $AA' > BB'$ we may decrease AA' by decreasing α, thus increasing BB', and vice versa, leading to a common minimum when $AA' = BB'$, ie, when $a^2 + 2bd + \sqrt{2}ad(\cos\alpha + \sin\alpha) = b^2$. Note also that

$$AA'^2 + BB'^2 = 3b^2 - 2bd + 2d^2 + 3a^2 + \sqrt{2}a(3d\cos\alpha - d\sin\alpha + 4b\sin\alpha).$$

For any given d, the minimum of the previous expression is reached when $\sin\alpha$ is minimum, since $\sin\alpha \geq \cos\alpha$, and $b \geq d$; moreover, when the previous expression is minimum, then $AA' = BB'$ is clearly minimum, or the minimum is reached when $\alpha = \frac{\pi}{4}$ and $2d = b-a$, yielding

$$\max\{AA'^2, BB'^2\} \geq b^2 + a^2 + d^2 + 3ab = \frac{5(a+b)^2}{4} \geq 20,$$

or $\max\{AA', BB'\} \geq 2\sqrt{5}$, hence at least equal to the diameter of the circle. Hence two squares whose sidelengths add up to 4 or more cannot be fitted, even touching one another's perimeter, inside a circle with radius $\sqrt{5}$. The conclusion follows.

S142. Consider two concentric circles $C_1(O,R)$ and $C_2(O,\frac{R}{2})$. Prove that for each point A on the circumference of circle C_1 and for each point Ω inside the circle C_2 there are points B and C on the circumference of C_1 such that Ω is the center of the nine-point circle of triangle ABC.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote by H the symmetric of O with respect to Ω, and draw the circle with center A passing through H. This circle intesects C_1 at two points, which we will call P and Q. Draw now the perpendicular bisectors of HP and HQ, which clearly pass through A, and intersect each the circumcircle at another point, respectively B and C. Now, $\angle QCA = \angle ACP$ because $AP = AQ$, or since AC is the perpendicular bisector of HQ, CP passes through H and is perpendicular to AB, and similarly BQ passes through H and is perpendicular to AC. Hence H is the orthocenter of ABC, and its nine-point center is the midpoint of H and O, which is Ω, as desired.

Note that we have not only constructed triangle ABC inscribed in C_1 such that Ω is its nine-point center, but from the construction we may find additional information: since H is clearly inside the circumcircle of ABC, we conclude that ABC will always be acute. Note also that we are not restricted to Ω being inside C_2: the proposed construction is valid as long as the symmetric of Ω with respect to O is such that $AH < 2R$, ie, Ω may be any point inside circle C_3 with center at the midpoint of AO and radius R (this clearly includes the interior of circle C_2). Note finally that whenever Ω is on the boundary of C_2 but inside C_3 (this excludes only the symmetric with respect to O of the midpoint of OA), the symmetric H of O with respect to Ω is on C_1, and is one of the vertices of ABC, the third one being the symmetric of A with respect to O, ABC being rectangle at H; if Ω is outside C_2 but inside C_3, then ABC inscribed in C_1 and such that Ω is its nine-point center may be constructed, but it is obtuse.

S143. Let m and n be positive integers, $m < n$. Evaluate

$$\sum_{k=m+1}^{n} k(k^2-1^2)\cdots(k^2-m^2).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author. Denoting the sum by $S_{m,n}$ we have

$$\begin{aligned}(2m+2)S_{m,n} &= \sum_{k=m+1}^{n} k(k^2-1^2)\cdots(k^2-m^2)[(k+m+1)-(k-m-1)] \\ &= \sum_{k=m+1}^{n} [(k-m)...k...(k+m+1)-(k-m-1)...k...(k+m)],\end{aligned}$$

which telescopes to $(n-m)...n...(n+m+1)$. Hence

$$S_{m,n} = \frac{1}{2(m+1)} \cdot \frac{(n+m+1)!}{(n-m-1)!}.$$

Second solution by Michel Bataille, France

Let $S_{m,n}$ denote the given sum. We will show that

$$S_{m,n} = (2m+1)!\binom{m+n+1}{2m+2} = \frac{1}{2m+2}\cdot\frac{(n+m+1)!}{(n-(m+1))!}.$$

We have

$$\begin{aligned}S_{m,n} &= \sum_{k=m+1}^{n} ((k-m)(k-(m-1))\cdots(k-1)k(k+1)\cdots(k+m)) \\ &= \sum_{k=m+1}^{n} \frac{(k+m)!}{(k-(m+1))!} = ((2m+1)!)\sum_{k=m+1}^{n}\binom{m+k}{2m+1}.\end{aligned}$$

Now, for nonnegative integers p, q, we have

$$\begin{aligned}\binom{p}{p}+\binom{p+1}{p}+\cdots+\binom{p+q}{p} &= \binom{p+1}{p+1}+\binom{p+1}{p}+\cdots+\binom{p+q}{p} \\ &= \binom{p+2}{p+1}+\binom{p+2}{p}+\cdots+\binom{p+q}{p} \\ &= \ldots\ldots\ldots\ldots\ldots \\ &= \binom{p+q}{p+1}+\binom{p+q}{p} = \binom{p+q+1}{p+1}.\end{aligned}$$

It follows that

$$\sum_{k=m+1}^{n} \binom{m+k}{2m+1} = \binom{m+n+1}{2m+2}$$

and so

$$S_{m,n} = (2m{+}1)!\binom{m+n+1}{2m+2} = (2m{+}1)!\frac{(m+n+1)!}{(2m+2)!(n-m-1)!} = \frac{1}{2m+2}\cdot\frac{(n+m+1)!}{(n-(m+1))!}.$$

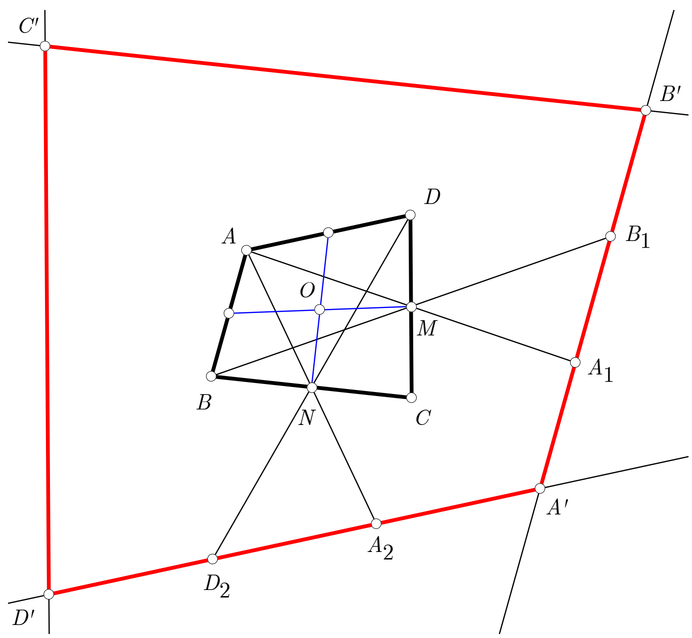
Also solved by G. C. Greubel, Newport News, USA; Raul A. Simon,Chile; Arkady Alt, San Jose, California, USA; Dmitri Skjorshammer, Harvey Mudd College, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S144. Let $ABCD$ be a quadrilateral. Consider the reflection of each of the lines AB, BC, CD, DA on the respective midpoints of the opposite sides CD, DA, AB, BC. Prove that these four lines bound a quadrilateral $A'B'C'D'$ homothetic with $ABCD$ and find the ratio and center of the homothety.

Proposed by Francisco Javier García Capitán and Juan Bosco Romero Márquez

First solution by Ercole Suppa, Teramo, Italy

Consider a system of coordinates with origin in the centroid O of $ABCD$ and denote by $\overrightarrow{X}$ the vector from O to X. Let A_1, B_1 be the reflections of A, B on the midpoint M of CD and let A_2, D_2 be the reflections of A, D on the midpoint N of BC, as shown in figure.



We clearly have $\overrightarrow{M} = \frac{\overrightarrow{C}+\overrightarrow{D}}{2}$, $\overrightarrow{N} = \frac{\overrightarrow{B}+\overrightarrow{C}}{2}$, hence

$$\overrightarrow{A_1} = \overrightarrow{C} + \overrightarrow{D} - \overrightarrow{A}, \qquad \overrightarrow{B_1} = \overrightarrow{C} + \overrightarrow{D} - \overrightarrow{B} \tag{1}$$

$$\overrightarrow{A_2} = \overrightarrow{B} + \overrightarrow{C} - \overrightarrow{A}, \qquad \overrightarrow{D_2} = \overrightarrow{B} + \overrightarrow{C} - \overrightarrow{D} \tag{2}$$

Since A' lies on the lines A_1B_1 and A_2D_2 there are suitable real numbers t, u such that

$$\overrightarrow{A'} = \overrightarrow{A_1} + t\left(\overrightarrow{B_1} - \overrightarrow{A_1}\right) = \overrightarrow{A_2} + u\left(\overrightarrow{D_2} - \overrightarrow{A_2}\right)$$

and using (1) and (2) we get

$$\overrightarrow{C} + \overrightarrow{D} - \overrightarrow{A} + t\left(\overrightarrow{A} - \overrightarrow{B}\right) = \overrightarrow{B} + \overrightarrow{C} - \overrightarrow{A} + u\left(\overrightarrow{A} - \overrightarrow{D}\right) \qquad \Rightarrow$$

$$(t-u)\overrightarrow{A} - (t+1)\overrightarrow{B} + (u+1)\overrightarrow{D} = 0$$

Since the vectors $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{D}$ are linearly independent we obtain

$$t = u = -1 \qquad \Rightarrow \qquad \overrightarrow{A'} = \overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D} - 2\overrightarrow{A} \tag{3}$$

From (3), taking into account that $\overrightarrow{O} = \frac{1}{4}\left(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D}\right)$, it follows that

$$\overrightarrow{A'} + 3 \cdot \overrightarrow{A} = 4 \cdot \overrightarrow{O} \qquad \Leftrightarrow \qquad \overrightarrow{OA'} = -3 \cdot \overrightarrow{OA}$$

and this implies that A' is the image of A under the homotety of center O and ratio $k = -3$. In similar way we can prove that B', C', D' are respectively the correspondents of B, C, D in the homotety with center O and ratio $k = -3$.

Therefore the quadrilateral $A'B'C'D'$ is the image of $ABCD$ under the homotety of center O and ratio $k = -3$. □

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Any two points X, Y, respectively on lines AB, DA, satisfy $\vec{AX} = \rho\vec{AB}$ and $\vec{AY} = \rho\vec{AD}$ for some reals ρ, κ. The midpoint M of CD clearly satisfies $\vec{AM} = \frac{1}{2}\vec{AC} + \frac{1}{2}\vec{AD}$, or the symmetric X' of X with respect to M is such that $\vec{AX'} + \rho\vec{AB} = \vec{AC} + \vec{AD}$. Similarly, the symmetric Y' of Y with respect to the midpoint of BC satisfies $\vec{AY'} + \kappa\vec{AD} = \vec{AB} + \vec{AC}$. Calling A' the point where these two lines meet, κ and ρ for A' must be such that $\vec{AD} - \rho\vec{AB} = \vec{AB} - \kappa\vec{AD}$, or $\kappa = \rho = -1$, and $\vec{AA'} = \vec{AB} + \vec{AC} + \vec{AD}$. Adding $4\vec{OA}$ to both sides, where O is the barycenter of $ABCD$, we find $3\vec{OA} + \vec{OA'} = \vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = \vec{0}$, and clearly $\vec{OA'} = -3\vec{OA}$, and similarly for the other three vertices of both quadrilaterals. We conclude that $ABCD$ and $A'B'C'D'$ are homothetic, the radio and center of the homothety are respectively -3 and the barycenter of $ABCD$, which is also the barycenter of $A'B'C'D'$.

Also solved by Raul A. Simon,Chile; Miguel Amengual Covas.

Undergraduate problems

U139. Find the least interval containing all values of the expression

$$E(x,y,z)=\frac{x}{x+2y}+\frac{y}{y+2z}+\frac{z}{z+2x}.$$

Proposed by Dorin Andrica, "Babeş-Bolyai" University, Cluj-Napoca, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that, whenever $x=0$, $y=2u-2$, $z=2-u$, for $u\neq 1$ and $u\neq 2$, then the expression takes the value $0+\frac{2u-2}{2u-2+4-2u}+1=u$, or any real value except for 1 and 2 may be obtained in this way. Note next that the expression takes value 1 when $x=y=z\neq 0$. Note finally that, when $x=\frac{\sqrt{17}-1}{2}$, $y=-4$ and $z=1$, the expression takes the value

$$\frac{\sqrt{17}-1}{\sqrt{17}-17}+2+\frac{1}{\sqrt{17}}=2.$$

Therefore, the expression takes all real values.

Also solved by Arkady Alt, San Jose, California, USA; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Ercole Suppa, Teramo, Italy.

U140. Let $(a_n)_{n\geq 1}$ a decreasing sequence of positive real numbers. Let

$$s_n = a_1 + a_2 + \ldots + a_n,$$

and

$$b_n = \frac{1}{a_{n+1}} - \frac{1}{a_n},$$

for all $n \geq 1$. Prove that if the sequence $(s_n)_{n\geq 1}$ is convergent, then the sequence $(b_n)_{n\geq 1}$ is unbounded.

Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Buzau, Romania

First solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

It is well known that if a decreasing sequence of positive terms $(a_n)_{n\geq 1}$ is such that $s_n = a_1 + \ldots + a_n$ tends to a limit, then $\lim_{n\to\infty} na_n = 0$ that is $na_n < \varepsilon$ for any $n \geq n_\varepsilon$ (ε may be choose as small as we need). Thus we have $\frac{1}{a_{N+1}} \geq \frac{N+1}{\varepsilon}$. Now let's suppose $b_n > 0$ bounded that is $0 < b_n \leq B$. We have $\frac{1}{a_{n+1}} \leq \frac{1}{a_n} + B$ yielding

$$\sum_{n=1}^{N} \frac{1}{a_{n+1}} \leq \sum_{n=1}^{N} \frac{1}{a_n} + BN \qquad \text{or} \qquad \frac{1}{a_{N+1}} < \frac{1}{a_1} + BN$$

but this contradicts $\frac{1}{a_{N+1}} \geq \frac{N+1}{\varepsilon}$ as soon as $\varepsilon < \frac{a_1(N+1)}{1+a_1NB}$ completing the proof.

Second solution by Arkady Alt, San Jose, California, USA

Suppose that there is $M > 0$ such that $b_n < M$. Then

$$\frac{1}{a_{n+1}} - \frac{1}{a_1} = \sum_{k=1}^{n} \left(\frac{1}{a_{k+1}} - \frac{1}{a_k} \right) = \sum_{k=1}^{n} b_k < nM$$

is equivalent to

$$\frac{1}{a_{n+1}} < nM + \frac{1}{a_1}$$

equivalent to

$$a_{n+1} > \frac{a_1}{a_1Mn + 1}, n \geq 1.$$

Let $K := \min\left\{a_1, \frac{1}{M}\right\}$ then $\frac{a_1}{a_1Mn+1} \geq \frac{K}{n+1}$. Furthermore $a_1n + a_1 \geq KMa_1 + K$ if and only if $a_1(1-KM) + (a_1 - K) \geq 0$.

Thus, $a_n > \frac{K}{n}, n \geq 2$ and, therefore, $s_n = a_1 + a_2 + ... + a_n > a_1 - 1 + h_n$, where $h_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ is a divergent harmonic sequence. This contradiction proves that $(b_n)_{n\geq 1}$ is unbounded.

Also solved by Michel Bataille, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U141. Find all pairs (x, y) of positive integers such that $13^x + 3 = y^2$.

Proposed by Andrea Munaro, Universit degli Studi di Trento, Italy

Solution by Andrea Munaro, Universit degli Studi di Trento, Italy

We have $(4-\sqrt{3})^x(4+\sqrt{3})^x = 13^x = (y-\sqrt{3})(y+\sqrt{3})$. It is easy to see that $\mathbb{Z}[\sqrt{3}]$ is a Euclidean domain with the norm N given by

$$N(a+b\sqrt{3}) = \left|a^2 - 3b^2\right|.$$

Hence $\mathbb{Z}[\sqrt{3}]$ is a PID and so a UFD.
Suppose there exists a prime $p \in \mathbb{Z}[\sqrt{3}]$ which divides both $y-\sqrt{3}$ and $y+\sqrt{3}$. Then

$$N(p) \mid N(y+\sqrt{3}) = \left|y^2 - 3\right| = 13^x.$$

On the other hand since $p \mid 2\sqrt{3}$, we have $N(p) \mid N(2\sqrt{3}) = 12$. Then $N(p) \mid (12, 13^x) = 1$ and so $N(p) = 1$, contradiction.
Hence $(y-\sqrt{3}, y+\sqrt{3}) = 1$ and so $y+\sqrt{3}$ is a x-power. In particular, since both $4-\sqrt{3}$ and $4+\sqrt{3}$ are primes then $(4+\sqrt{3})^x = y+\sqrt{3}$, which after comparing coefficients of $\sqrt{3}$ in both sides yields

$$1 = \sum \binom{x}{2k+1} 3^k 4^{x-(2k+1)} = x4^{x-1} + (\text{terms} \geq 1).$$

Therefore $x = 1$.

U142. Let $f : [0,1] \to \mathbb{R}$ be a continuously differentiable function. Prove that if $\int\limits_0^{\frac{1}{2}} f(x)dx = 0$ then

$$\int\limits_0^1 (f'(x))^2 dx \geq 12 \left(\int\limits_0^1 f(x)dx \right)^2 .$$

Proposed by Duong Viet Thong Faculty of Foundation, Nam Dinh University of Technology Education, Phu Nghia Road, Loc Ha Ward, Nam Dinh City, Vietnam

First solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Define the function $g(x) = \begin{cases} ax & 0 \leq x \leq 1/2 \\ -a(x-1) & 1/2 \leq x \leq 1 \end{cases}$ We have

$$\int_0^1 f(x)g'(x)dx = f(x)g(x)\Big|_0^1 - \int_0^1 f'(x)g(x)dx = -\int_0^1 f'(x)g(x)dx$$

but we have also

$$\int_0^1 f(x)g'(x)dx = a\int_0^{1/2} f(x)dx - a\int_{1/2}^1 f(x)dx = -a\int_0^1 f(x)dx$$

thanks to the condition on $f(x)$. Cauchy–Schwarz yields

$$\left(\int_0^1 f'(x)g(x)dx\right)^2 \leq \int_0^1 (f'(x))^2 dx \int_0^1 (g(x))^2 dx$$

thus

$$a^2 \left(\int_0^1 f(x)dx\right)^2 \leq a^2 \left(\int_0^1 (f'(x))^2 dx\right) \cdot \left(\int_0^{1/2} x^2 dx + \int_{1/2}^1 (x-1)^2 dx\right)$$

and finally

$$\left(\int_0^1 f(x)dx\right)^2 \leq \frac{1}{12}\int_0^1 (f'(x))^2 dx.$$

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Define $g(x) = \frac{f(x)+f(1-x)}{2}$ and $h(x) = \frac{f(x)-f(1-x)}{2}$. Clearly $g(x) = g(1-x)$, $h(x) = h(1-x)$ and $f(x) = g(x) + h(x)$, and moreover $f'(x) = g'(x) + h'(x)$, where $g'(x) = -g'(1-x)$ and $h'(x) = h'(1-x)$, leading to $\int_0^1 g'(x)h'(x)dx = 0$. Note also that we may define

$$K = \int_0^{\frac{1}{2}} g(x)dx = \int_{\frac{1}{2}}^1 g(x)dx = -\int_0^{\frac{1}{2}} h(x)dx = \int_{\frac{1}{2}}^1 h(x)dx.$$

The proposed inequality thus rewrites as

$$\int_0^{\frac{1}{2}} (g'(x))^2 dx + \int_0^{\frac{1}{2}} (h'(x))^2 dx \geq 24K^2,$$

where $g'\left(\frac{1}{2}\right) = 0$ and $h\left(\frac{1}{2}\right) = 0$. Clearly $(g'(x))^2 \geq 0$ for all $x \in \left[0, \frac{1}{2}\right]$, with equality iff $g(x) = 2K$ for all $x \in [0,1]$, while using the Cauchy-Schwarz inequality,

$$\frac{1}{24}\int_0^{\frac{1}{2}} (h'(x))^2 dx = \left(\int_0^{\frac{1}{2}} x^2 dx\right)\left(\int_0^{\frac{1}{2}} (h'(x))^2 dx\right) \geq \left(\int_0^{\frac{1}{2}} x h'(x) dx\right)^2 =,$$

$$= \left(\frac{1}{2}h\left(\frac{1}{2}\right) - 0 \cdot h(0) - \int_0^{\frac{1}{2}} h(x)dx\right)^2 = K^2,$$

with equality iff $h'(x) = \rho x$ for all $x \in \left[0, \frac{1}{2}\right]$ and some real constant ρ. The conclusion follows, and equality holds iff, for all $x \in \left[0, \frac{1}{2}\right)$,

$$h(x) = -\int_x^{\frac{1}{2}} h'(t)dt = -\rho \int_x^{\frac{1}{2}} t dt = \rho \frac{4x^2 - 1}{8}.$$

Moreover, it must hold that

$$-K = \int_0^{\frac{1}{2}} h(x)dx = \rho \int_0^{\frac{1}{2}} \frac{4x^2-1}{8} dx = -\frac{\rho}{24},$$

ie, equality holds in the proposed inequality iff $f(x) = 12Kx^2 - K$ for all $x \in \left[0, \frac{1}{2}\right]$ and $f(x) = -7K + 24Kx - 12Kx^2$ for all $x \in \left[\frac{1}{2}, 1\right]$, for some real constant K, where we have used the relation between $h(x)$ and $h(1-x)$.

U143. For a positive integer $n > 1$, determine

$$\lim_{x\to 0} \frac{\sin^2(x)\sin^2(nx)}{n^2\sin^2(x)-\sin^2(nx)}.$$

Proposed by N. Javier Buitrago A., Universidad Nacional de Colombia

First solution by Ercole Suppa, Teramo, Italy

First of all, let us rewrite the function in the following simpler form by using the well known identity $\sin^2 x = \frac{1-\cos x}{2}$:

$$\frac{\sin^2(x)\sin^2(nx)}{n^2\sin^2(x)-\sin^2(nx)} = \frac{1-\cos x-\cos(nx)+\cos x\cos(nx)}{2n^2-2n^2\cos x-2+2\cos(nx)}$$

Now, a repeated application of L'Hospital's Rule gives the result

$$\lim_{x\to 0}\frac{1-\cos x-\cos(nx)+\cos x\cos(nx)}{2n^2-2n^2\cos x-2+2\cos(nx)} =$$
$$= \lim_{x\to 0}\frac{\sin x+n\sin(nx)-\cos(nx)\sin x-n\cos x\sin(nx)}{2n^2\sin x-2n\sin(nx)} =$$
$$= \lim_{x\to 0}\frac{\cos x+n^2\cos(nx)-\cos x\cos(nx)-n^2\cos x\cos(nx)+2n\sin x\sin(nx)}{2n^2\cos x-2n^2\cos(nx)} =$$
$$= \lim_{x\to 0}\frac{-\sin x+\left(1+3n^2\right)\cos(nx)\sin x-n^3\sin(nx)+\left(3n+n^3\right)\cos x\sin(nx)}{-2n^2\sin x+2n^3\sin(nx)} =$$
$$= \lim_{x\to 0}\frac{-\cos x+\left(1+6n^2+n^4\right)\cos x\cos(nx)-n^4\cos(nx)-4\left(n+n^3\right)\sin x\sin(nx)}{-2n^2\cos x+2n^4\cos(nx)} =$$
$$= \frac{6n^2}{2n^4-2n^2} = \frac{3}{n^2-1}$$

□

Second solution by Michel Bataille, France

We shall use $u(x) \sim v(x)$ to mean $\lim_{x\to 0}\frac{u(x)}{v(x)} = 1$ and $o(x^n)$ to denote any function of the form $x^n\varepsilon(x)$ where $\lim_{x\to 0}\varepsilon(x) = 0$.
From the well-known $\sin x \sim x$, we deduce $\sin^2(x)\sin^2(nx) \sim n^2x^4$ (1).
On the other hand, from $\sin x = x - \frac{x^3}{6} + o(x^4)$, we first obtain

$$\sin^2 x = x^2 - \frac{x^4}{3} + o(x^4)$$

and then

$$n^2\sin^2(x)-\sin^2(nx) = n^2x^2\left(1-\frac{x^2}{3}+o(x^2)\right)-n^2x^2\left(1-\frac{n^2x^2}{3}+o(x^2)\right)$$
$$= n^2x^2\left(\frac{(n^2-1)x^2}{3}+o(x^2)\right).$$

Thus,

$$n^2\sin^2(x)-\sin^2(nx)\sim\frac{n^2(n^2-1)x^4}{3} \qquad (2).$$

Finally, (1) and (2) readily yield

$$\lim_{x\to 0}\frac{\sin^2(x)\sin^2(nx)}{n^2\sin^2(x)-\sin^2(nx)}=\frac{3}{n^2-1}.$$

Third solution by John Mangual

Let L denote the limit:

$$L=\lim_{x\to 0}\frac{\sin^2 x\sin^2 nx}{n^2\sin^2 x-\sin^2 nx}$$

Then taking reciprocals the fractions simplify a bit:

$$1/L=\lim_{x\to 0}\frac{n^2}{\sin^2 nx}-\frac{1}{\sin x^2}$$

We can estimate $\sin x$ to third order $x-x^3/6$ without getting divergences:

$$\begin{aligned} 1/L &= \lim_{x\to 0}\frac{n^2}{\left(nx-\frac{n^3x^3}{6}\right)^2}-\frac{1}{\left(x-\frac{x^3}{6}\right)^2} \\ &= \lim_{x\to 0}\frac{1}{\left(x-\frac{n^2x^3}{6}\right)^2}-\frac{1}{\left(x-\frac{x^3}{6}\right)^2} \\ &= \lim_{x\to 0}\frac{1}{x^2}\left[\frac{1}{\left(1-\frac{n^2x^2}{6}\right)^2}-\frac{1}{\left(1-\frac{x^2}{6}\right)^2}\right] \end{aligned}$$

Finally, use the square of the geometric series formula: $1/(1-x)^2=1+2x+3x^2+\dots$ up to second order:

$$1/L=\lim_{x\to 0}\frac{1}{x^2}\left[\left(1+\frac{n^2x^2}{3}\right)-\left(1-\frac{x^2}{3}\right)=\frac{n^2-1}{3}\right]$$

Therefore, the resulting limit is $L=3/(n^2-1)$.

Also solved by Arkady Alt, San Jose, California, USA; Bedri Hajrizi, Albania; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U144. Let F be the set of all continuous functions $f : [0, \infty) \to [0, \infty)$ satisfying the relation

$$f\left(\int_0^x f(t)dt\right) = \int_0^x f(t)dt$$

for all $x \in [0, \infty)$.

a) Prove that F has infinitely many elements.

b) Find all convex functions f in the set F.

Proposed by Mihai Piticari, "Dragos-Voda" National College, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

a) Define $f(x) = x$ for $x \in [0, a]$, $f(x) = 2a - x$ for $x \in [a, 2a]$ and $f(x) = 0$ for all $x \geq 2a$. Clearly $f(x)$ is continuous, and the maximum value of $\int_0^x f(t)dt$ is a^2, or $f \in F$ as long as $a^2 \leq a$, ie, as long as $a \leq 1$. The set F contains therefore at least all the functions $f(x)$ thus defined for the infinite values $a \in [0, 1]$.

b) Define $g(x) = \int_0^x f(t)dt$. Clearly, $f(g(x)) = g(x)$, $g(x)$ is continuous, and $g(0) = 0$. If $g(x)$ is not bounded, by the intermediate value theorem, for any positive real x a real y exists such that $g(y) = x$, or $f(x) = f(g(y)) = g(y) = x$, and $f(x) = x$ for all $x \in [0, \infty)$. Note that $f(x) = x$ is convex (not strictly). Assume now that $g(x)$ is bounded; then, $S = \sup_{x\in[0,\infty)} g(x)$ exists, and by the intermediate value theorem $f(x) = x$ for all $x \in [0, S)$. Assume that $y > x$ exists such that $f(y) < y$, then point $\left(\frac{S}{2}, \frac{S}{2}\right)$ is on the graph of $f(x)$ and above the line through points $(0, 0)$ and $(y, f(y))$, which are also on the graph of $f(x)$, and $f(x)$ would be concave; contradiction, hence $f(x) \geq x$ for all $x > S$, and $g(x)$ is not bounded, contradiction. Hence if $g(x)$ is bounded and its supreme S is positive, either $f(x) = x$, or $f \notin F$, or f is not convex. Moreover, if $S = 0$, clearly $f(x) = 0$ for all x, and again we obtain a (non strictly) convex function in F. We conclude that the only convex functions in F are $f(x) = x$ for all $x \in [0, \infty)$ and $f(x) = 0$ for all $x \in [0, \infty)$; neither is strictly convex, but any other function in F is necessarily concave.

Olympiad problems

O139. Through point M on the circle circumscribed to the acute triangle ABC, draw parallels to the sides BC, CA, AB which interesect the circle the second time at points A', B', C' ($A' \in \overset{\frown}{BC}, B' \in \overset{\frown}{CA}, C' \in \overset{\frown}{AB}$). If $\{D\} = A'B' \cap BC, \{E\} = A'B' \cap CA, \{F\} = B'C' \cap CA, \{D'\} = B'C' \cap AB, \{E'\} = A'C' \cap AB, \{F'\} = A'C' \cap BC$, prove that the lines DD', EE', FF' are concurrent.

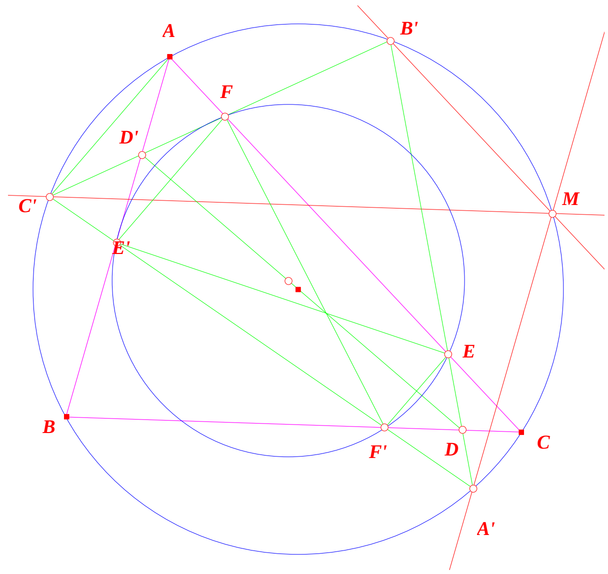
Proposed by Cătălin Barbu, Colegiul "Vasile Alecsandri", Bacau Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let P, Q, R be the respective points where the parallels through M to BC, CA, AB intersect the circumcircle of ABC. Clearly $\angle PMQ = \angle ACB$, $\angle QMR = \angle BAC$ and $\angle RMP = \angle CBA$, or since they are all chords of the same circle, $PQ = AB$, $QR = BC$ and $RP = CA$. We conclude that triangles ABC and PQR are equal, or there is a diameter of the circumcircle of ABC with respect to which ABC and PQR are symmetric; denote by r this diameter. Clearly, this diameter intersects ABC in two points, which are also points of PQR, wlog D and D' (they may be others according to the position of M on the circumcircle, but we may rename the points D, E, F, D', E', F' rotating them cyclically without altering the problem). Now, points E and F' are clearly symmetric with respect to DD', and so are F and E', or $EF'E'R$ is an isosceles trapezoid whose parallel sides have perpendicular bisector DD', hence its diagonals EE' and FF' intersect on DD'. The conclusion follows.

Second solution by Tarik Adnan Moon, Bangladesh

At first we prove that $\triangle ABC \cong C'B'A'$. Here $\angle A'B'C' = \angle C'MA' = \angle ABC$, as MC', MA' are parallel to BC, AB respectively. Analogously $\angle B'A'C' = \angle BCA$. So, Both of the triangles have the same circumcircle, hence they have equal circumradius. We get, $A'C' = 2R \sin B' = 2R \sin B = AC$. Thus we conclude that $\triangle ABC \cong C'B'A'$.



Now we prove that $EFE'F'$ is a cyclic isosceles trapezoid. Here $AC = AC'$. So, $AC'A'C$ is a cyclic isosceles trapezoid. Thus $\angle C'AC = \angle AC'A'$. Also $AC' \ll A'C$. From the congruence of the triangles we deduce that $\angle E'AF = \angle E'C'F$, so $AC'E'F$ is a cyclic quadrilateral. Analogously $ECA'F'$ is a cyclic quadrilateral. We have, $\angle C'AF = \angle FE'F'$ and $\angle C'AF = \pi - \angle ECA' = \pi - \angle EF'E'$ So, $EF' \parallel FE'$. Also, $\angle C'AF = \pi - \angle CA'F' = \angle F'AC$. So, $EFE'F'$ is a cyclic isosceles trapezoid. We also notice that $\triangle D'E'F$ and $\triangle DEF'$ are isosceles triangles erected on $E'F$ and EF'. So, the result follows from symmetry.

O140. Let n be a positive integer and let $x_k \in [-1,1]$, $1 \le k \le 2n$, such that $\sum_{k=1}^{2n} x_k$ is an odd integer. Prove that

$$1 \le \sum_{k=1}^{2n} |x_k| \le 2n-1.$$

Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Buzau, Romania

First solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

We begin with the r.h.s. If $x_k \equiv 1$ we would have

$$\sum_{k=1}^{2n} x_k = \sum_{k=1}^{2n} |x_k| = 2n$$

but this is excluded by the hypotheses so at most we can have

$$\sum_{k=1}^{2n} |x_k| \le 2n-1$$

L.h.s. Since $\sum_{k=1}^{2n} x_k = 1+2p$ suppose that $p \ge 0$, p integer. The inequality follows by

$$\sum_{k=1}^{2n} |x_k| \ge \sum_{k=1}^{2n} x_k = 1+2p \ge 1.$$

Now let p negative integer. We have

$$\sum_{k=1}^{2n} |x_k| \ge \sum_{k=1}^{2n} (-x_k) = -2p-1 \ge 1.$$

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume wlog that $x_1+x_2+\cdots+x_{2n}$ is positive, since changing each x_k into $-x_k$ does not alter the problem. Clearly, $|x_1|+|x_2|+\cdots+|x_{2n}| \ge x_1+x_2+\cdots+x_{2n} \ge 1$, with equality iff all x_k are positive and their sum is 1.

Assume now that $|x_1|+|x_2|+\cdots+|x_{2n}| > 2n-1$, and call S_+ the sum of all positive x_k, and S_- the sum of all negative x_k in absolute value. Then, $S_+ + S_- > 2n-1$ and $S_+ - S_- = 2m+1$ is a positive odd integer. Hence, $S_+ > n+m$ and $S_- > n-m-1$. Since each x_k is at most 1 in absolute value, there are at least $n+m+1$ positive x_k's and at least $n-m$ negative x_k's, yielding at least $2n+1$ x_k's, absurd. The conclusion follows.

O141. Let S_n be the set of all $3n$-digit numbers consisting of n 1^s,n 2^s , and n 5^s. Prove that for each n there are at least 4^{n-1} numbers in S_n that can be written as sum of the cubes of some $n+1$ distinct positive integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that $125 = 5^3$ and $512 = 8^3$. Note also that $152 = 125+27 = 5^3+3^3$, which makes the result true for $n = 1$. Assume now that the result is true for $n-1$, and let N be any of the at least 4^{n-2} $3n-3$-digit numbers formed by $n-1$ 1^{s}, $n-1$ 2^{s} and $n-1$ 5^{s} which may be written as the sum of n distinct positive cubes, ie, positive integers $x_1, x_2, \ldots, x_n$ exist such that $N = x_1^3+x_2^3+\cdots+x_n^3$. Note now that $(5\cdot 10^{n-1})^3+x_1^3+x_2^3+\cdots+x_n^3$, $(8\cdot 10^{n-1})^3+x_1^3+x_2^3+\cdots+x_n^3$, $(10x_1)^3+(10x_2)^3+\cdots+(10x_n)^3+5^3$ and $(10x_1)^3+(10x_2)^3+\cdots+(10x_n)^3+8^3$ are $3n$-digit numbers formed by n 1^{s}, n 2^{s} and n 5^{s} (they have actually the same digits as N, with either 125 or 512, either in front or behind), and may clearly be written as the sum of $n+1$ distinct positive cubes, since $10x_k$ cannot be equal to either 5 or 8, and $5\cdot 10^{n-1}$ and $8\cdot 10^{n-1}$ are necessarily larger than any x_k, since their cubes are larger than N itself. Hence, if there are at least 4^{n-2} numbers that satisfy the desired property for $n-1$, there are at least $4\cdot 4^{n-2} = 4^{n-1}$ numbers that satisfy the desired property for n. By induction, the conclusion follows.

O142. If m is a positive integer show that 5^m+3 has neither a prime divisor of the form $p=30k+11$ nor of the form $p=30k-1$.

Proposed by Andrea Munaro, Universita degli Studi di Trento, Italy

First solution by Andrea Munaro, Universita degli Studi di Trento, Italy

In the following denote by $\left(\frac{a}{p}\right)$ the Legendre symbol.

First suppose that m is even. Then $\left(5^{\frac{m}{2}}\right)^2 \equiv -3 \pmod p$, and so $\left(\frac{-3}{p}\right)=1$. But $\left(\frac{-3}{p}\right)=\left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)$ and by quadratic reciprocity law we have

$$\left(\frac{3}{p}\right)\left(\frac{p}{3}\right)=(-1)^{\frac{p-1}{2}}.$$

Also $\left(\frac{p}{3}\right)\equiv p\equiv -1 \pmod 3$ and $\left(\frac{-1}{p}\right)=(-1)^{\frac{p-1}{2}}$. Then $\left(\frac{-3}{p}\right)=-1$, absurd.

If m is odd we have $\left(5^{\frac{m+1}{2}}\right)^2\equiv -15 \pmod p$ and so $\left(\frac{-15}{p}\right)=1$. But $\left(\frac{-15}{p}\right)=\left(\frac{-3}{p}\right)\left(\frac{5}{p}\right)=-\left(\frac{5}{p}\right)$, and $\left(\frac{p}{5}\right)\equiv p^2\equiv 1 \pmod 5$. Again by quadratic reciprocity law we have

$$\left(\frac{5}{p}\right)\left(\frac{p}{5}\right)=(-1)^{p-1}=1,$$

and so $\left(\frac{5}{p}\right)=1$. Finally $\left(\frac{-15}{p}\right)=-1$, absurd.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Euler, in his paper "Theoremata circa divisores numerorum in hac forma $paa\pm qbb$ contentorum", states that an integer of the form a^2+3b^2, where a,b are relatively prime integers, can only have prime divisors 2, 3, or of the form c^2+3d^2, where c,d must clearly be relatively prime positive integers of opposite parity, and 3 does not divide c. He also states that an integer of the form $3a^2+5b^2$, where a,b are relatively prime integers, can only have prime divisors 2, 3, 5, or of one of the forms c^2+15d^2 or $3e^2+5f^2$, where clearly c,d are relatively prime integers of opposite parity, e,f are relatively prime integers of opposite parity, 3 does not divide c or f, and 5 does not divide c or e.

Since $c^2\equiv 1 \pmod 3$ for any c not divisible by 3, it follows that a^2+3b^2 may only have divisors 2, 3, or congruent to $1,7,13,19,25$ modulus 30, hence neither of the form $30k+11$ nor $30k-1$. Now, if m is even, $5^m+3=a^2+3b^2$, where $a=5^{\frac{m}{2}}$ and $b=1$.

Since $c^2\equiv 1 \pmod 3$ and $c^2\equiv \pm 1 \pmod 5$ for any c not divisible by 3 or 5, it follows that c^2+15d^2 is congruent to 1 or 4 modulus 15, hence neither of the form $30k+11$ nor $30k-1$. Similarly, since $e^2 \equiv \pm 1 \pmod 5$ if 5 does not divide e and $f^2 \equiv 1 \pmod 3$ if 3 does not divide f, if follows again that $3e^2+5f^2$ is congruent to 1 or 4 modulus 15. Now, if m is odd, $5^m+3 = 5b^2+3a^2$, where $b = 5^{\frac{m-1}{2}}$ and $a = 1$.

The conclusion follows.

Note: We include a proof of the first statement, ie, an integer of the form a^2+3b^2, where a, b are relatively prime positive integers, can only have prime divisors 2, 3, or odd primes of the form c^2+3d^2, where c, d are relatively prime positive integers.

Proof: If $a = 3b'$ for some integer b', call $b = a'$, yielding $a^2+3b^2 = 3(a'^2+3b'^2)$, proceeding this way until a is not a multiple of 3. It thus suffices to prove that a^2+3b^2, where a is not a multiple of 3, has only prime divisors 2 and odd primes of the form c^2+3d^2.

If a^2+3b^2 is even, and since a, b are relatively prime, then a, b are both odd, hence a^2+3b^2 is a multiple of 4 since odd perfect squares leave remainder 1 when divided by 8. Note that 4 may be written as c^2+3d^2 with $c, d = 1$.

Assume now that a^2+3b^2 is divisible by $p = c^2+3d^2$, where c, d are relatively prime positive integers, and either p is an odd prime other than 3, or $p = 4$ and clearly $c = d = 1$. In the latter case, and since a, b must both be odd, note that either $ad+bc = a+b$ and $ac-3bd = a-3b$ are both multiples of 4, or $ad-bc = a-b$ and $ac+3bd = a+3b$ are both multiple of 4. In the former case,

$$(ac+3bd)(ac-3bd) = a^2c^2-9b^2d^2 = (a^2+3b^2)c^2-3b^2(c^2+3d^2),$$

where c^2+3d^2 is an odd prime that divides a^2+3b^2, and hence it must also divide either $ac+3bd$ or $ac-3bd$. Moreover, $c(ac \pm 3bd) - a(c^2+3d^2) = 3d(bc \mp ad)$, and since c^2+3d^2 clearly cannot divide $3d$, it must divide either $bc-ad$ or $bc+ad$, respectively. It follows that either $e = \frac{ac-3bd}{c^2+3d^2}$ and $f = \frac{bc+ad}{c^2+3d^2}$ are both integers, or $e = \frac{ac+3bd}{c^2+3d^2}$ and $f = \frac{ad-bc}{c^2+3d^2}$ are both integers, while in either case, $(c^2+3d^2)(e^2+3f^2) = a^2+3b^2$. Therefore, if a^2+3b^2 is divisible by an odd prime of the form $c^2+3d^2 \neq 3$, or if it is even and hence divisible by $c^2+3d^2 = 4$, then $\frac{a^2+3b^2}{c^2+3d^2}$ is also of the form e^2+3f^2.

Assume finally that a^2+3b^2, not multiple of 2 or 3, is the least composite number of this form that is not divisible by numbers of the form c^2+3d^2, and call p the smallest of its prime divisors (clearly odd). Write $a = mp \pm a'$, $b = np \pm b'$, where m, n are integers and wlog $|a'|, |b'| < \frac{p}{2}$ are also integers. Clearly $a^3+3b^2 = (m^2p+3n^2p \pm 2a'm \pm 6b'n)p + a'^2 + b'^2$, or p divides $a'^2+3b'^2$. Since a factor not of the form c^2+3d^2 divides $a'^2+3b'^2 \leq a^2+3b^2$, then $a' = a$ and $b' = b$, or $\frac{a^2+3b^2}{p}$ is a divisor of a^2+3b^2 greater than 1 and smaller than

p, absurd. Thus all composite numbers of the form a^2+3b^2, not multiple of 2 or 3, have some prime divisor of the form $p=c^2+3d^2$, hence dividing by p, we find a new number of the form a^2+3b^2, which is therefore, either prime, or divisible by another prime of this same form. An infinite descent of this type is impossible, since in each division we reduce the number of prime factors by 1, hence for all relatively prime a,b, all prime divisors of a^2+3b^2 other than 2 or 3 must be primes of the form c^2+3d^2.

The proof of the second statement is similar, except that numbers of both forms $3a^2+5b^2$ and a^2+15b^2 must be mixed along the proof (note that $(3c^2+5d^2)(3e^2+5f^2)=(3cd\pm 5df)^2+15(de\mp cf)^2$, that $(3c^2+5d^2)(e^2+15f^2)=3(ce\pm 5df)^2+5(de\mp 3cf)^2$, and that $(c^2+15d^2)(e^2+15f^2)=(ce\pm 15df)^2+15(de\mp cf)^2$). Also, it must be used that if $3a^2+5b^2$ or a^2+15b^2 are even and a,b are relatively prime (hence both odd), it follows that 8 divides them, and $8=3c^2+5d^2$, with $c=d=1$. This, together with divisions by 3 or 5, reduces the problem to finding the prime divisors of numbers of the form $3a^2+5b^2$ and a^2+15b^2 that are not divisible by 2, 3 or 5. A similar process as the one followed for the first statement yields again that these prime divisors can only be of one of the forms c^2+15d^2 or $3c^2+5d^2$.

Third solution by Srinath.R, Chennai, India

Assume that there is a prime number p of the form $30k+11$ or $30k-1$ dividing 5^m+3. Before proceeding to the main solution, I will present the following two theorems and give the source of the theorems at the end .

(a) -3 is a quadratic residue modulo p if and only if $p\equiv 1 \bmod 6$ where p is an odd prime number .

(b) 5 is a quadratic residue modulo p if and only if $p\equiv \pm 1 \bmod 10$ where p is an odd prime number .

Lets take two cases and prove separately for $p=30k+11$ and $p=30k-1$.

Case 1- $p=30k+11$

We have $5^m\equiv -3 \bmod p$ .Since m is a positive integer ,it can either be even or odd . So if $m=2k$ where $k\in\mathbb{N}$,we have $5^{2k}\equiv -3 \bmod p$. Since 5^{2k} is a perfect square it implies that -3 is a qaudratic residue modulo p. Thus $\left(\frac{-3}{p}\right)=1$ where $\left(\frac{a}{p}\right)$ denotes the Legendre Symbol of a with respect to p . By Theorem 1 , -3 can be a quadratic residue modulo p if and only if $p\equiv 1 \bmod 6$. But $p\equiv 11 \bmod 30 \Rightarrow p\equiv 5 \bmod 6$,so p is a quadratic non residue mod p,thus we get a contradiction. So m is odd .Let $m=2k-1$ where $k\in\mathbb{N}$. So ,

$$5^{2k-1}\equiv -3 \bmod p \Rightarrow 5^{2k}\equiv -15 \bmod p \Rightarrow \left(\frac{-15}{p}\right)=1 \ (1)$$

Now since the Legendre symbol has the multiplicative property ,

$$\left(\frac{-15}{p}\right) = \left(\frac{5}{p}\right)\left(\frac{-3}{p}\right)$$

Since $p \equiv 11 \bmod 30 \Rightarrow p \equiv 1 \bmod 10$. So by theorem 2 , 5 is a quadratic residue mod p . Thus ,$\left(\frac{5}{p}\right) = 1$ and we have already proved $\left(\frac{-3}{p}\right) = -1$. So from both these conditions it follows that $\left(\frac{-15}{p}\right) = -1$,contradicting (1) . This completes the proof of the theorem when $p = 30k + 11$.

Case 2 -$p = 30k - 1$

We can proceed as the proof for Case 1 . We have $5^m \equiv -3 \bmod p$. Since m is a positive integer ,it can either be even or odd . So if $m = 2k$ where $k \in \mathbb{N}$,we have $5^{2k} \equiv -3 \bmod p$. Since 5^{2k} is a perfect square it implies that -3 is a qaudratic residue modulo p. Thus $\left(\frac{-3}{p}\right) = 1$. By Theorem 1 , -3 can be a quadratic residue modulo p if and only if $p \equiv 1 \bmod 6$. But $p \equiv 29 \bmod 30 \Rightarrow p \equiv 5 \bmod 6$,so p is a quadratic non residue mod p ,thus we get a contradiction. So m is odd .Let $m = 2k - 1$ where $k \in \mathbb{N}$. So ,

$$5^{2k-1} \equiv -3 \bmod p \Rightarrow 5^{2k} \equiv -15 \bmod p \Rightarrow \left(\frac{-15}{p}\right) = 1(2)$$

As the legendre symbol has the multiplicative property

$$\left(\frac{-15}{p}\right) = \left(\frac{5}{p}\right)\left(\frac{-3}{p}\right)$$

Since $p \equiv 29 \bmod 30 \Rightarrow p \equiv -1 \bmod 10$. So by theorem 2 , 5 is a quadratic residue mod p . Thus ,$\left(\frac{5}{p}\right) = 1$ and we have already proved $\left(\frac{-3}{p}\right) = -1$ as $p \equiv 5 \bmod 6$. So from both these conditions it follows that $\left(\frac{-15}{p}\right) = -1$,contradicting (2) . This completes the proof of the theorem when $p = 30k - 1$ .So the proof for the problem is complete .

Source for the theorems:

The theorem 1 and theorem 2 can be found in the file Quadratic Congruences written by Dusan Djukic,author of IMO compendium ,as part of the article for the International Mathematics Olympiad ,at the website

http://www.imomath.com/tekstkut/quadcong_ddj.pdf

O143. Let $ABCDEF$ be a convex hexagon such that the sum of the distances of each interior point to the six sides is equal to the sum of the distances between the midpoints of AB and DE, BC and EF, and CD and FA. Prove that $ABCDEF$ is cyclic.

Proposed by Nairi Sedrakyan, Armenia

Solution by Nairi Sedrakyan, Armenia

Let AD and CF, CF and BE, BE and AD intersect at points M, N, P, respectively. Let O be the incenter of triangle MNP. In the case when M, N, P overlap, O is point M. Let $A_1, B_1, C_1, D_1, E_1, F_1$ be the midpoints of sides AB, BC, CD, DE, EF, and FA. Let $d_a, d_b, d_c, d_d, d_e, d_f$ the distance from point O to the lines AB, BC, CD, DE, EF, FA. Finally, let $A_0, B_0, C_0, D_0, E_0, F_0$ be the intersection of lines MO, NO, PO with lines AB, BC, CD, DE, EF, FA. Note that $A_0D_0 \geq d_a + d_b,\ B_0E_0 \geq d_b + d_e,\ C_0F_0 \geq d_c + d_f$ hence

$$A_0D_0 + B_0E_0 + C_0F_0 \geq d_a + d_b + d_c + d_d + d_e + d_f. \qquad (1)$$

On the other hand we have $\angle A_1A_0D_0 \geq 90^\circ$ and $\angle D_1D_0A_0 \geq 90^\circ$, because in any triangle the bisector is between the median and the height if they are all drawn from the same vertex. Thus the projection of segment A_1D_1 onto A_0D_0 contains A_0D_0, and this implies that $A_1D_1 \geq A_0D_0$. Analoguously we get $B_1E_1, B_0E_0, C_1F_1 \geq C_0F_0$. From the last 3 inequalities and (1) we get

$$A_1D_1 + B_1E_1 + C_1F_1 \geq d_a + d_b + d_c + d_d + d_e + d_f. \qquad (2)$$

The inequality (2) becomes equality due to the give3n condition. Note that the last happens only when points $A_1, B_1, C_1, D_1, E_1, F_1$ are identical to points $A_0, B_0, C_0, D_0, E_0, F_0$. Hence we get that lines B_0E_0, C_0F_0, A_0D_0 are the medians and perpendiculars of segments BC and EF, CD and AF, DE and AB, respectively. In conclusion point O is at the same distance from points A, B, C, D, E, F and hence $ABCDEF$ can be inscribed in a circle.

O144. Find all positive integers a, b, c such that $(2^a - 1)(3^b - 1) = c!$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Claim: For any positive integer m, $\phi(3^m) = 2 \cdot 3^{m-1}$ is the least exponent k such that 3^m divides $2^k - 1$.

Proof: The result is easily checked by inspection for $m = 1, 2$. By Euler-Fermat's theorem, $3^m | 2^{\phi(3^m)} - 1$. Let now k be the least positive integer such that $3^m | 2^k - 1$ for some $m \geq 3$, and write $\phi(3^m) = qk + r$, where $r \in \{0, 1, \ldots, k - 1\}$. Clearly, $2^{\phi(3^m)} = (2^k)^q \cdot 2^r \equiv 2^r \equiv 1 \pmod{3^m}$, and $3^m | 2^r - 1$, or $r = 0$ since k is minimum, and k divides $\phi(3^m)$. If $k < \phi(3^m)$, then either $k = 3^u$ (impossible, since $2^k \equiv -1 \pmod 3$ for any odd k), or $k = 2 \cdot 3^u$. In this latter case,

$$2^{2 \cdot 3^u} - 1 = (2^6 - 1)(2^{12} + 2^6 + 1)(2^{36} + 2^{18} + 1) \ldots (2^{4 \cdot 3^{u-1}} + 2^{2 \cdot 3^{u-1}} + 1).$$

Note that since $2^6 \equiv 1 \pmod 9$, all brackets in the RHS except for the first one are congruent 3 modulus 9, or they are divisible by 3 but not by 9. Since $2^6 - 1$ is divisible by 9 but not by 27, the highest exponent of 3 that divides the RHS is thus $u - 1 + 2 = u + 1$, or $u \geq m - 1$. The conclusion follows.

Assume that some solution is possible with $c \geq 9$. The exponent of 3 that divides $c!$ is $\lfloor \frac{c}{3} \rfloor + \lfloor \frac{c}{3^2} \rfloor + \lfloor \frac{c}{3^3} \rfloor + \ldots$, clearly larger than $\frac{c}{3}$ when $c \geq 9$ since $\lfloor \frac{c}{3^2} \rfloor \geq 1$. By the claim, $a > 2 \cdot 3^{\frac{c}{3} - 1}$ and $\ln(2^a - 1) > 3^{\frac{c}{3} - 1}$. Since $\ln(c!) < \ln(c^c) = c \ln(c)$, then there will be no solution as long as $c > 3 + 3 \log_3(c \ln c)$. Assume now that there is a solution for $c = 15$. Clearly, it must be $4 < \log_3(15 \ln 15)$, or $\ln 15 > \frac{81}{15} > 5$, which is false. There is therefore no solution for $c = 15$. Moreover, since c grows faster than $3 + 3 \log_3(c \ln c)$ (just compare their derivatives), then there is no solution for $c \geq 15$. Assume now that there is a solution for $9 \leq c \leq 14$. Clearly 3^4 divides $2^a - 1$, or $a \geq 54$. Now, $2^{54} - 1 > 2^{44} = 16^6 \cdot 8^4 \cdot 4^4 > 14!$, or there is no solution for $c \geq 9$. Assume next that there is a solution for $c = 8$. Therefore, 3^2 divides $2^a - 1$, or 6 divides a. If $a \geq 12$, then $2^a - 1 \geq 4095 > 315 = \frac{8!}{2^7}$ (because $2^a - 1$ cannot be divided by 2), and there is no solution; hence $a = 6$, or $3^b - 1 = \frac{8!}{63} = 640$. But 641 is not a power of 3, and we conclude that there is no solution for $c \geq 8$. Assume finally that there is a solution for $c = 6$; as before, $a = 6$, absurd, since $2^6 - 1$ is divisible by 7 but 6! is not. Clearly $c = 1$ cannot have any solution since $3^b - 1$ is at least 2, or there may be solutions only for $c = 2, 3, 4, 5, 7$. There are actually solutions for all these values of c:

$2! = 2 = (2^1 - 1)(3^1 - 1)$, $\quad 3! = 6 = (2^2 - 1)(3^1 - 1)$, $\quad 4! = 24 = (2^2 - 1)(3^2 - 1)$,

$$5! = 120 = (2^4 - 1)(3^2 - 1), \qquad 7! = 5040 = (2^6 - 1)(3^4 - 1).$$

The possible values for (a, b, c) are then $(1, 1, 2)$, $(2, 1, 3)$, $(2, 2, 4)$, $(4, 2, 5)$ and $(6, 4, 7)$. There may not be any other solutions for the given values of c; the case $c = 7$ immediately forces $a = 6$ as before, wherefrom $b = 4$ results by direct calculation; the case $c = 5$ forces $3|2^a - 1$ and $8|3^b - 1$, the second condition resulting in b even, $b \geq 4$ being ruled out since $3 \cdot 80 = 240 > 5!$, wherefrom $2^a - 1 = 15$; finally, all cases where $c \leq 4$ result in $c! = 2^u 3^v$, which clearly force $2^u = 3^b - 1$ and $3^v = 2^a - 1$, with unique solutions.

Also solved by Paolo Leonetti, Bocconi University, Italy; Raul A. Simon, Chile; Vahagn Aslanyan, Yerevan, Armenia.