Solution to problem J614

Let a, b, c be real numbers such that $abc = 1$. Prove that

$$\frac{a}{\sqrt{1+b^2c+bc^2}} + \frac{b}{\sqrt{1+c^2a+ca^2}} + \frac{c}{\sqrt{1+a^2b+ab^2}} \geq \frac{a+b+c}{3}$$

Proof Using $abc = 1$ the inequality is

$$\sum_{\text{cyc}} \frac{a}{\sqrt{1+\frac{b}{a}+\frac{c}{a}}} = \frac{a\sqrt{a}+b\sqrt{b}+c\sqrt{c}}{\sqrt{a+b+c}} \geq \frac{a+b+c}{\sqrt{3}}$$

that is

$$\sqrt{3}(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}) \geq (a+b+c)^{3/2}$$

This is power–means–inequality

$$\frac{(a^{3/2}+b^{3/2}+c^{3/2})^{2/3}}{3^{2/3}} \geq \frac{a+b+c}{3} \iff 3^{1/3}(a^{3/2}+b^{3/2}+c^{3/2})^{2/3} \geq a+b+c$$

and then by elevating to the fractional power $3/2$ we get the desired inequality

$$3^{1/2}(a^{3/2}+b^{3/2}+c^{3/2}) \geq (a+b+c)^{3/2}$$