*Originally problem 6 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2002.*

*There were eight solution submitted for this question, all essentially the same.*

By the Pythagorean Theorem we have

$$AC = \sqrt{BC^2 + AB^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

and

$$DC = AC - AD = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5}-1}{2}.$$

**CC108**. In an orthonormal system, the line with equation $y = 5x$ crosses the parabola with equation $y = x^2$ in point $A$. The perpendicular to $OA$ at $O$ intersects the parabola at $B$. What is the area of triangle $AOB$?

*Originally problem 20 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2009.*

*We received six correct solutions, and one incorrect solution. We present the solution of Titu Zvonaru.*

It is easy to deduce that $A(5, 25)$. The slope of $OB$ is $-1/5$. Solving the system $y = -\frac{1}{5}x$, $y = x^2$ we obtain $B(-\frac{1}{5}, \frac{1}{25})$.

It follows that $OA = \sqrt{5^2 + 25^2} = 5\sqrt{26}$, $OB = \sqrt{\frac{1}{5^2} + \frac{1}{25^2}} = \frac{\sqrt{26}}{25}$. Hence the area of the triangle is $AOB = \frac{OA \cdot OB}{2} = \frac{26}{10} = \frac{13}{5}$.

**CC109**. Let $E$ be the set of reals $x$ for which the two sides of the following equality are defined:

$$\cot 8x - \cot 27x = \frac{\sin kx}{\sin 8x \sin 27x}.$$

If this equality holds for all the elements of $E$, what is the value of $k$?

*Originally problem 21 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2009.*

*We received seven submitted solutions to this problem, one of which was incorrect and five were incomplete. We present the only correct solution by Paolo Perfetti modified by the editor.*

Note first that $E = \{x \in \mathbb{R} | x \neq \frac{m\pi}{8}$ and $x \neq \frac{m\pi}{27}$ for any $m \in \mathbb{Z}$. For $x \in E$, the given equality is equivalent to

$$\sin 8x \cdot \sin 27x(\cot 8x - \cot 27x) = \sin kx. \qquad (1)$$

We shall prove that the only value of $k$ for which (1) holds for all $x \in E$ is $k = 19$.

Since

$$\begin{aligned}\sin 8x \cdot \sin 27x(\cot 8x - \cot 27x) &= \sin 27x \cos 8x - \cos 27x \sin 8x \\ &= \sin(27x - 8x) = \sin 19x,\end{aligned}$$

$k = 19$ satisfies (1).

Next, suppose (1) holds for all $x \in E$ and some $k \in \mathbb{Z}$ with $k \neq 19$.

If $k = -19$, then from (1) we have $2\sin 19x = 0$ for all $x \in E$, which is false (for example, if $x = \frac{\pi}{38}$, then $x \in E$, but $\sin 19x = \sin\frac{\pi}{2} = 1 \neq 0$). Hence $k \neq -19$.

From (1), we also have

$$2\sin\left(\frac{19-k}{2}x\right)\cos\left(\frac{19+k}{2}x\right) = 0. \qquad (2)$$

Since $\sin\left(\frac{19-k}{2}x\right) = 0$ if and only if $\frac{19-k}{2}x = m\pi$ or $x = \frac{2m\pi}{19-k}$ and $\cos\left(\frac{19+k}{2}x\right) = 0$ if and only if $\frac{19+k}{2}x = (m+\frac{1}{2})\pi$ or $x = \frac{(2m+1)\pi}{19+k}$ for some $m \in \mathbb{Z}$, there must be some $x \in E$ that does not satisfy (2). (To be more precise, the set of all $x$ such that $x = \frac{2m\pi}{19-k}$ or $x = \frac{(2m+1)\pi}{19+k}$ for some $m \in \mathbb{Z}$ is countable while $E$ is clearly uncountable.) This is a contradiction and our proof is complete.

**CC110**. What is the number of real solutions to the equation:

$$|1 + x - |x - |1 - x||| = |-x - |x - 1||.$$

*Originally problem 26 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2009.*

*We have received four correct solutions and one incorrect submission. We present the solution by Henry Ricardo.*

We compute the left-hand side (LHS) and the right-hand side (RHS) on three intervals that cover the real number line.

*Case 1.* Suppose that $0 \leq x \leq 1$. Then

$$\text{RHS} = |-x - (1-x)| = |-x - 1 + x| = 1.$$

When $x \in [-, \frac{1}{2}]$,

$$|1 + x - |x - (1-x)|| = |1 + x - (1 - 2x)| = 3x$$

and when $x \in (\frac{1}{2}, 1]$,

$$|1 + x - |2x - 1|| = |1 + x - (2x - 1)| = |2 - x| = 2 - x$$

so that

$$\text{LHS} = \begin{cases} 3x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2 - x & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Copyright © Canadian Mathematical Society, 2015

*Solution 2, abridged version of the solution by the Missouri State University Problem Solving Group.*

Let $A = (0, 0)$, $B = (1, 0)$, $C = (0, 1)$, $M = (a, 0)$, and $N = (0, b)$, where $a$ and $b$ are rational and $0 < a < b < 1$. The equations of the lines $\overline{BN}$ and $\overline{CM}$ have rational coefficients, so the coordinates of $O$ are rational. The area of a triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ is

$$\frac{1}{2}\left|\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}\right|.$$

Therefore, the areas of $MBO$, $BCO$, $CNO$, and $AMON$ are all rational. By stretching the triangle $ABC$, the corresponding areas can be made to be integers. Since stretching does not alter the ratios $AM/MB$ and $AN/NC$, the configurations are not affinely equivalent for distinct choices of $a$ and $b$.

*Solution 3, by Titu Zvonaru.*

Let $a, b, m, n$ be positive integers and let $ABC$ be a triangle with $BC = 2a$ and $h_A = b(m + 1)(n + 1)(m + n + 1)$. Choose the points $M$ and $N$ on $\overline{AB}$ and $\overline{AC}$ such that

$$\frac{BM}{BA} = \frac{1}{m+1}, \quad \frac{CN}{CA} = \frac{1}{n+1}.$$

Denote by $[XY \dots Z]$ the area of the polygon $XY \dots Z$. Then

$$[BMC] = \frac{[ABC]}{m+1}, \quad [CNB] = \frac{[ABC]}{n+1}.$$

Suppose that $\overline{AO}$ intersects $\overline{BC}$ at $A'$. By Van Aubel's Theorem for Cevian triangles we obtain

$$\frac{AO}{OA'} = \frac{AM}{MB} + \frac{AN}{NC} = m + n$$

and therefore $OA' = AA'/(m + n + 1)$. It follows that

$$[BOC] = \frac{[ABC]}{m+n+1}.$$

Thus the areas $[ABC]$, $[BMC]$, $[CNB]$, and $[BOC]$ are all integers and by taking differences of these areas so are $[MBO]$, $[CNO]$, and $[AMON]$.

## 3916. *Proposed by Nathan Soedjak.*

Let $a, b, c$ be positive real numbers. Prove that

$$\left(\frac{ab}{c}\right)^2 + \left(\frac{bc}{a}\right)^2 + \left(\frac{ca}{b}\right)^2 \geq 3\left(\frac{ab+bc+ca}{a+b+c}\right)^2.$$

*There were 23 correct solutions, with two solutions from one solver, as well as a Maple verification. We present a sampling of the different approaches.*

Copyright © Canadian Mathematical Society, 2015

*Solution 1, by Mohammed Aassila.*

Note that $x^2 + y^2 + z^2 \geqslant xy + yz + zx$ and $(x + y + z)^2 \geqslant 3(xy + yz + zx)$ for real $x, y, z$. The left side of the inequality is not less than $b^2 + c^2 + a^2$. However

$$(a^2 + b^2 + c^2)(a + b + c)^2 \geqslant (ab + bc + ca)[3(ab + bc + ca)] = 3(ab + bc + ca)^2,$$

and the desired result follows.

*Solution 2, by Michel Bataille.*

By homogeneity, we may suppose that $a + b + c = 1$. The inequality is then equivalent to

$$(ab)^4 + (bc)^4 + (ca)^4 \geqslant 3(a^2b^2c^2)(ab + bc + ca)^2.$$

Observe that

$$\begin{aligned} x^4 + y^4 + z^4 &= \frac{1}{4}[(x^4 + x^4 + y^4 + z^4) + (x^4 + y^4 + y^4 + z^4) + (x^4 + y^4 + z^4 + z^4)] \\ &\geqslant x^2yz + xy^2z + xyz^2 = xyz(x + y + z), \end{aligned}$$

and $(x + y + z)^2 \geqslant 3(xy + yz + zx)$. Applying these inequalities leads to

$$\begin{aligned} (ab)^4 + (bc)^4 + (ca)^4 &= [(ab)^4 + (bc)^4 + (ca)^4][(a + b + c)^2] \\ &\geqslant [a^2b^2c^2(ab + bc + ca)][3(ab + bc + ca)] \\ &= 3(a^2b^2c^2)(ab + bc + ca)^2, \end{aligned}$$

as desired.

*Solution 3, by Dionne Bailey, Elsie Campbell, and Charles Dimminnie; Angel Plaza; Cao Minh Quang; and Edmund Swylan, independently.*

Since $x^2 + y^2 + z^2 \geqslant xy + yz + zx$,

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = \frac{(ab)^2 + (bc)^2 + (ca)^2}{abc} \geqslant \frac{abc(a + b + c)}{abc} = a + b + c.$$

Using either the convexity of the function $x^2$ or the inequality of the root-mean-square and arithmetic mean, we find that

$$\begin{aligned} \left(\frac{ab}{c}\right)^2 + \left(\frac{bc}{a}\right)^2 + \left(\frac{ca}{b}\right)^2 &\geqslant \frac{1}{3}\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right)^2 \\ &\geqslant \frac{1}{3}(a + b + c)^2 = \frac{1}{3}\frac{(a + b + c)^4}{(a + b + c)^2} \\ &\geqslant \frac{[3(ab + bc + ca)]^2}{3(a + b + c)^2} = 3\left(\frac{ab + bc + ca}{a + b + c}\right)^2. \end{aligned}$$

*Solution 4, by Paolo Perfetti.*

Since

$$\frac{1}{4}\left(\frac{x^2y^2}{z^2}+\frac{x^2y^2}{z^2}+\frac{y^2z^2}{x^2}+\frac{z^2x^2}{y^2}\right)\geqslant xy$$

by the arithmetic-geometric means inequality, we can follow the strategy of Solution 2 to obtain

$$\left(\frac{ab}{c}\right)^2+\left(\frac{bc}{a}\right)^2+\left(\frac{ca}{b}\right)^2\geqslant ab+bc+ca=\frac{3(ab+bc+ca)^2}{3(ab+bc+ca)}\geqslant\frac{3(ab+bc+ca)^2}{(a+b+c)^2}$$

as desired.

*Solution 5 by Henry Ricardo.*

We have

$$\begin{aligned}\left(\frac{ab}{c}\right)^2+\left(\frac{bc}{a}\right)^2+\left(\frac{ca}{b}\right)^2&=\frac{1}{2}\left[a^2\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+b^2\left(\frac{a^2}{c^2}+\frac{c^2}{a^2}\right)+c^2\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)\right]\\&\geqslant a^2+b^2+c^2\\&\geqslant\frac{(a+b+c)^2}{3}\\&\geqslant 3\left(\frac{ab+bc+ca}{a+b+c}\right)^2.\end{aligned}$$

## **3917**. *Proposed by Peter Y. Woo.*

Given a circle $Z$, its center $O$, and a point $A$ on $Z$, with only a long unmarked ruler, and no compass, can you draw:

i) points $B$,$C$ and $D$ on $Z$ so that $ABCD$ is a square?

ii) the square $AOBA'$?

iii) the points $B, W'', W$ and $W'$ on $Z$ such that angles $AOB$, $AOW''$, $AOW$ and $AOW'$ are $90°, 60°, 45°$ and $30°$?

*There were five correct solutions to this problem. We feature the one by the Missouri State University Problem Solving Group.*

We need the following basic construction: Given three collinear points $A, B, C$ such that $AB = BC$ and a point $P$ not on $\overleftrightarrow{AC}$, we want to construct a line through $P$ parallel to $\overleftrightarrow{AC}$. To do this, we choose a point $Q$ on the ray $\overrightarrow{AP}$ such that $P$ is between $A$ and $Q$. Denote the intersection of $\overleftrightarrow{BQ}$ and $\overleftrightarrow{CP}$ by $R$ and denote the intersection of $\overleftrightarrow{AR}$ and $\overleftrightarrow{QC}$ by $S$. We claim that $\overleftrightarrow{PS}$ is the line we seek. By Ceva's theorem,

$$\frac{AB}{BC}\cdot\frac{CS}{SQ}\cdot\frac{QP}{PA}=1,$$

**3920**. *Proposed by Alina Sîntămărian.*

Evaluate

$$\sum_{n=0}^{\infty} \frac{16n^2+20n+7}{(4n+2)!}.$$

*There were 15 submitted solutions for this problem, 14 of which were correct. We present three solutions, representative of the two main solution methods utilized together with one variant.*

*Solution 1, by the AN-anduud Problem Solving Group.*

Consider the following two power series,

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}, \quad \text{and} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}.$$

Hence, we have

$$\sin 1 = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n+1)!} = \sum_{n=0}^{\infty} \left( \frac{1}{(4n+1)!} - \frac{1}{(4n+3)!} \right),$$

and

$$e = \sum_{n=1}^{\infty} \frac{1}{n!}.$$

Using the above considerations, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{16n^2+20n+7}{(4n+2)!} &= \sum_{n=0}^{\infty} \frac{(4n+2)(4n+1)+2(4n+2)+1}{(4n+2)!} \\ &= \sum_{n=0}^{\infty} \left( \frac{1}{(4n)!} + \frac{2}{(4n+1)!} + \frac{1}{(4n+2)!} \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \left( \frac{1}{(4n+1)!} - \frac{1}{(4n+3)!} \right) \\ &= e + \sin 1. \end{aligned}$$

*Solution 2, by the group of Dionne Bailey, Elsie Campbell, and Charles Diminnie.*

To begin, we note that for $n \geqslant 0$,

$$\begin{aligned} \frac{16n^2+20n+7}{(4n+2)!} &= \frac{(4n+2)\,(4n+1)+2\,(4n+2)+1}{(4n+2)!} \\ &= \frac{1}{(4n)!} + \frac{2}{(4n+1)!} + \frac{1}{(4n+2)!}, \end{aligned}$$

and hence,

$$\sum_{n=0}^{\infty} \frac{16n^2+20n+7}{(4n+2)!} = \sum_{n=0}^{\infty} \frac{1}{(4n)!} + 2\sum_{n=0}^{\infty} \frac{1}{(4n+1)!} + \sum_{n=0}^{\infty} \frac{1}{(4n+2)!}$$

(since the Ratio Test easily confirms that each of the three series on the right converges).

The remainder of this solution depends on the following known series:

$$\sin 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}, \quad \cos 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!},$$
$$\sinh 1 = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}, \quad \cosh 1 = \sum_{k=0}^{\infty} \frac{1}{(2k)!}.$$

Since we have

$$(-1)^k + 1 = \begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} \quad \text{and} \quad (-1)^{k+1} + 1 = \begin{cases} 0 & \text{if } k \text{ is even} \\ 2 & \text{if } k \text{ is odd} \end{cases},$$

we obtain:

$$\sin 1 + \sinh 1 = \sum_{k=0}^{\infty} \frac{(-1)^k+1}{(2k+1)!} = \sum_{n=0}^{\infty} \frac{2}{[2(2n)+1]!} = 2\sum_{n=0}^{\infty} \frac{1}{(4n+1)!},$$
$$\cos 1 + \cosh 1 = \sum_{k=0}^{\infty} \frac{(-1)^k+1}{(2k)!} = \sum_{n=0}^{\infty} \frac{2}{[2(2n)]!} = 2\sum_{n=0}^{\infty} \frac{1}{(4n)!},$$
$$-\cos 1 + \cosh 1 = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}+1}{(2k)!} = \sum_{n=0}^{\infty} \frac{2}{[2(2n+1)]!} = 2\sum_{n=0}^{\infty} \frac{1}{(4n+2)!}.$$

Therefore, we obtain,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{16n^2+20n+7}{(4n+2)!} &= \frac{\cos 1 + \cosh 1}{2} + (\sin 1 + \sinh 1) + \frac{-\cos 1 + \cosh 1}{2} \\ &= \sin 1 + \sinh 1 + \cosh 1 \\ &= \sin 1 + \frac{e - e^{-1}}{2} + \frac{e + e^{-1}}{2} \\ &= \sin 1 + e.\end{aligned}$$

*Solution 3, by Paolo Perfetti.*

First, we have:

$$\frac{16n^2+20n+7}{(4n+2)!} = \frac{1}{(4n)!} + \frac{2}{(4n+1)!} + \frac{1}{(4n+2)!}.$$

Copyright © Canadian Mathematical Society, 2015

Let

$$f(x)=\sum_{n=0}^{\infty}\frac{x^{4n}}{(4n)!},$$

so that we obtain:

$$f'(x)=\sum_{n=1}^{\infty}\frac{x^{4n-1}}{(4n-1)!},\quad f''(x)=\sum_{n=1}^{\infty}\frac{x^{4n-2}}{(4n-2)!},\quad f'''(x)=\sum_{n=1}^{\infty}\frac{x^{4n-2}}{(4n-2)!}$$

$$f^{iv}(x)=\sum_{n=1}^{\infty}\frac{x^{4n-4}}{(4n-4)!}=\sum_{n=0}^{\infty}\frac{x^{4n}}{(4n)!}=f(x).$$

Thus $f(x)$ satisfies $f^{iv}(x)=f(x),\ f(0)=1,\ f'(0)=0,\ f''(0)=0,\ f'''(0)=0,$ whose unique solution is $f(x)=\frac{1}{2}\cosh x+\frac{1}{2}\cos x$. Evaluating, we get

$$f(1)=\frac{1}{2}\cosh 1+\frac{1}{2}\cos 1=\sum_{n=0}^{\infty}\frac{1}{(4n)!}.$$

Moreover, if we define

$$g(x)=\sum_{n=0}^{\infty}\frac{x^{4n+1}}{(4n+1)!},$$

we get $g(1)=\sum_{n=0}^{\infty}\frac{1}{(4n+1)!}$ and $g'(x)=f(x),\ g(0)=0$. This implies

$$g(x)=\frac{1}{2}\sinh x+\frac{1}{2}\sin x,\quad g(1)=\frac{1}{2}\sinh 1+\frac{1}{2}\sin 1.$$

Finally, defining

$$h(x)=\sum_{n=0}^{\infty}\frac{x^{4n+2}}{(4n+2)!},$$

we get $h(1)=\sum_{n=0}^{\infty}\frac{1}{(4n+2)!}$ and $h'(x)=g(x),\ h(0)=0$. This implies

$$h(x)=\frac{1}{2}\cosh x-\frac{1}{2}\cos x,\quad h(1)=\frac{1}{2}\cosh 1-\frac{1}{2}\cos 1.$$

Summing up the terms, we obtain

$$f(1)+2g(1)+h(1)=e+\sin 1.$$

*Editor's Comment.* The presented solutions illustrate three techniques: rearrange the summations wisely to get a simple expression, rearrange the summations and then recall other atypical power series that make things work, and solve a couple of DEs to avoid having to work too much with power series. Wagon commented that the sum can be explicitly computed when the numerator is an arbitrary quadratic in $n$.