We received six correct submissions and one incorrect submission. We present the solution by Henry Ricardo.

Property 3 holds if and only if the sum of the integer's digits is divisible by 3.

Thus if k, $k+1$, $k+2$, and $k+3$ are four consecutive digits used to construct a number satisfying our requirements, then their sum, $4k+6$, must be divisible by 3. This forces the smallest digit, k, to be a multiple of 3. Eliminating $k=9$, we are left with $k=0$, 3 or 6.

Since a four-digit number cannot begin with 0, there are $3 \cdot 3!$ permutations of the digits 0, 1, 2, 3 that form a number satisfying our conditions. Then there are 4! permutations of 3, 4, 5, 6 and 4! permutations of 6, 7, 8, 9 giving us a total of 66 positive integers satisfying all three properties.

CC93. If $x, y, z > 0$ and $xyz = 1$, find the range of all possible values of

$$\frac{x^3+y^3+z^3-x^{-3}-y^{-3}-z^{-3}}{x+y+z-x^{-1}-y^{-1}-z^{-1}}.$$

Originally SMT 2012, problem 11 of Team Test.

We received two correct solutions and one incorrect solution. We present the solution of Šefket Arslanagić, modified slightly by the editor.

Let

$$M = \frac{x^3+y^3+z^3-x^{-3}-y^{-3}-z^{-3}}{x+y+z-x^{-1}-y^{-1}-z^{-1}}.$$

Since $xyz = 1$, we have

$$M = \frac{(x^3-1)(y^3-1)(z^3-1)}{(x-1)(y-1)(z-1)} = (x^2+x+1)(y^2+y+1)(z^2+z+1).$$

By the AM-GM inequality,

$$x^2+x+1 \geq 3x, \quad y^2+y+1 \geq 3y, \quad \text{and} \quad z^2+z+1 \geq 3z,$$

so that $M \geq 27xyz = 27$. Equality occurs in this last inequality if and only if $x = y = z = 1$, which is excluded. The range is thus a subset of $(27, \infty)$. To see that the range is in fact all of $(27, \infty)$, note for instance that if $z = 1$, then

$$M = 3(x^2+x+1)\left(\frac{1}{x^2}+\frac{1}{x}+1\right) = \frac{3(x^2+x+1)^2}{x^2}.$$

The function $f(x) = \frac{3(x^2+x+1)^2}{x^2}$ is continuous on its domain $(0, \infty)$, and $\lim_{x\to 1} f(x) = 27$, while $\lim_{x\to\infty} f(x) = \infty$.

CC94. If $\log_2 x, (1+\log_4 x)$ and $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible values of x.

Originally 2009 Euclid Contest, problem 9a.

We received three correct submissions and six incorrect solutions (most people here did not properly account for the case when the second and third terms are 0, which is not allowed for a geometric sequence). We present the solution by Paolo Perfetti.

Let a be the common ratio of our geometric sequence and let $b = \log_2 x$. We can rewrite the terms of our sequence as $\log_2 x, 1 + \frac{1}{2}\log_2 x, \frac{2}{3} + \frac{1}{3}\log_2 x$. Then:

$$1 + \frac{1}{2}b = ab, \qquad \frac{2}{3} + \frac{1}{3}b = a^2 b.$$

Solving for (a, b) yields $(2/3, 6)$ and $(0, -2)$. The solution $(0, -2)$ is inadmissable, since the ratio of a geometric sequence cannot be 0. Thus $\log_2 x = 6$ so $x = 64$.

CC95. Positive integers x, y, z satisfy $xy + z = 160$. Determine the smallest possible value of $x + yz$.

Originally American Regions Mathematics League Competition 2013, problem 5 (Team).

We received four correct submissions and two incorrect submissions. We present the solution by Alina Sîntămărian.

The smallest possible value of $x + yz$ is 50.

Let $a = x + yz$. We analyze the following cases.

- $\boxed{y = 1}$ Then $z = 160 - x$ and $a = x + 160 - x = 160$.
- $\boxed{y = 2}$ Because $2x + z = 160$, it follows that $x \le 79$. Then

  $$a = x + 2(160 - 2x) \quad \Longrightarrow \quad x = \frac{2 \cdot 160 - a}{3} \le 79 \quad \Longrightarrow \quad a \ge 83.$$

  For $x = 79$ and $z = 2$ we have $a = 83$.
- $\boxed{y = 3}$ Because $3x + z = 160$, it follows that $x \le 53$. Then, from $a = x + 3(160 - 3x)$ we get that $a \ge 56$. For $x = 53$ and $z = 1$ we have $a = 56$.
- $\boxed{y = 4}$ Because $4x + z = 160$, it follows that $x \le 39$. Then, from $a = x + 4(160 - 4x)$ we get that $a \ge 55$. For $x = 39$ and $z = 4$ we have $a = 55$.
- $\boxed{y = 5}$ Because $5x + z = 160$, it follows that $x \le 31$. Then, from $a = x + 5(160 - 5x)$ we get that $a \ge 56$. For $x = 31$ and $z = 5$ we have $a = 56$.
- $\boxed{y = 6}$ Because $6x + z = 160$, it follows that $x \le 26$. Then, from $a = x + 6(160 - 6x)$ we get that $a \ge 50$. For $x = 26$ and $z = 4$, we have $a = 50$.
- $\boxed{y = 7}$ Because $7x + z = 160$, it follows that $x \le 22$. Then, from $a = x + 7(160 - 7x)$ we get that $a \ge 64$. For $x = 22$ and $z = 6$ we have $a = 64$.