Poly. U.; WEN LIU (student), Kent State U.; JERRY MINKUS, San Francisco, CA; RUTHVEN MURGATROYD, Albany, OR; R. S. TIBERIO, Wellesley High School, Wellesley, MA; MICHAEL VOWE, Therwil, Switzerland; ALBERT WHITCOMB, Castle Shannon, PA; YAJUN YANG, SUNY-Farmingdale; CHRISTIAN YANKOV, Eastern Connecticut State U.; HONGBIAO ZENG, Fort Hays State U.; and the proposer.

Limit of the sum of some improper integrals

859. *Proposed by José Luis Díaz-Barrero, Universítat Politècnica de Cataluña, Barcelona, Spain.*

Calculate

$$\lim_{n\to\infty}\left\{\frac{1}{n^4}\left(\sum_{k=1}^{n}k^2\int_k^{k+1}\ln[(x-k)^x(k+1-x)^x]dx\right)\right\}.$$

Solution by Jeffrey Groah, Montgomery College, Conroe, TX; Eugene Herman, Grinnell College, Grinnell, IA; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, independently.

Using integration by parts, we can establish that

$$\int x\ln(ax+b)\,dx = \frac{1}{2}\left[x^2\ln(ax+b)-\frac{1}{2}x^2+\frac{b}{a}x-\frac{b^2}{a^2}\ln(ax+b)\right].$$

The improper integral

$$\begin{aligned}
&\int_k^{k+1}\ln[(x-k)^x(k+1-x)^x]\,dx\\
&\quad=\lim_{\delta\to0^+}\int_{k+\delta}^{k+1-\delta}x\ln(x-k)+x\ln(k+1-x)\,dx\\
&\quad=\frac{1}{2}\lim_{\delta\to0^+}\{x^2\ln(x-k)+x^2\ln(k+1-x)-x^2-(2k+1)x\\
&\qquad-k^2\ln(x-k)-(k+1)^2\ln(k+1-x)\}\big|_{k+\delta}^{k+1-\delta}\\
&\quad=\frac{1}{2}\lim_{\delta\to0^+}\{[(k+1-\delta)^2\ln(1-\delta)+(k+1-\delta)^2\ln\delta\\
&\qquad-(k+1-\delta)^2-(2k+1)(k+1-\delta)-k^2\ln(1-\delta)\\
&\qquad-(k+1)^2\ln\delta)]-[(k+\delta)^2\ln\delta+(k+\delta)^2\ln(1-\delta)-(k+\delta)^2\\
&\qquad-(2k+1)(k+\delta)-k^2\ln\delta-(k+1)^2\ln(1-\delta)]\}\\
&\quad=\frac{1}{2}\lim_{\delta\to0^+}[((k+1-\delta)^2-(k+1)^2)\ln\delta-((k+\delta)^2-k^2)\ln\delta]-(2k+1)\\
&\quad=-(2k+1).
\end{aligned}$$

Using the fact that $\lim_{\delta \to 0^+} \delta \ln \delta = 0$, we have

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \frac{1}{n^4} \left(\sum_{k=1}^{n} k^2 \int_k^{k+1} \ln[(x-k)^x (k+1-x)^x] dx \right) \right\} \\ &\quad = -\lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^2 (2k+1) \\ &\quad = -\lim_{n\to\infty} \frac{1}{n^4} \left(\frac{1}{2} n^4 + \frac{4}{3} n^3 + n^2 + \frac{1}{6} n \right) = -\frac{1}{2}. \end{aligned}$$

Also solved by Michael Andreoli, Miami Dade C.; M. Reza Akhlaghi, Big Sandy Comm. and Tech. C.; Michel Bataille, Rouen, France; Joan Bell, Northeastern State U., OK; Brian Bradie, Christopher Newport U.; Minh Can, Irvine Valley C.; Timothy Chung (student), Foothill C.; Gordon Crandall, LaGuardia C.C.; Robert Crise, Crafton Hills C.; Chip Curtis, Missouri Southern State U.; Richard Daquila, Muskingum C.; Bill Dunn III, Montgomery C.; Neil Eklund, Centre C.; Habib Far, Montgomery C.; Michael Goldenberg and Mark Kaplan (jointly), Baltimore Poly. Inst.; Simone Lamont and Erick Timmons (students) with Farley Mawyer (jointly), York C. (CUNY); Elias Lampakis, Kiparissia, Greece; Kee-Wai Lau, Hong Kong, China; Leo Livshutz, Truman C.; David Lovit (student), U. of Washington; Kim McInturff, Santa Barbara, CA; Jerry Metzger and Thomas Richards (jointly), U. of North Dakota; Yozo Mikata, Schenectady, NY; Missouri State University Problem-Solving Group, Missouri State U.; Darryl Nester, Bluffton University; Ossama Saleh and Terry Walters (jointly), U. of Tennessee at Chattanooga; Nigel Salts, St. Francis C.; William Seaman, Albright C.; Peter Simone, U. of Nebraska Medical Center; Nora Thornber, Raritan Valley C.C.; Michael Vowe, Therwil, Switzerland; Mike Winders, Worcester State U.; Yajun Yang, SUNY-Farmingdale; Lanee Young, Fort Hays State U.; and the proposer. Two incorrect solutions were received.

Editor's Note. Neil Eklund of Centre College generalized this problem as follows: For every nonnegative integer j and all real numbers a and b,

$$\lim_{n\to\infty} \frac{1}{n^4} \left(\sum_{k=1}^{n} k^{3-j} \int_k^{k+1} \ln[(x-k)^{ax^j} (k+1-x)^{bx^j}]\, dx \right) = -\frac{a+b}{4}.$$

Conditions for cyclic hexagon

860. *Proposed by David Finn and Herb Bailey, Rose-Hulman Institute of Technology, Terre Haute, IN.*

Let I be the incenter and r the inradius of triangle ABC, and let X be its excenter. Let R be the exradius corresponding to the circle tangent to side BC and also tangent to extensions of sides AB and AC. Let D and E be the respective points of contact of the incircle and excircle with BC. Show that if ID is extended to F so that $|DF| = R$, and XE is extended to G so that $|EG| = r$, then the hexagon $BGICFX$ is cyclic.

Solution by Habib Far, Montgomery College, Conroe, TX; John Heuver, Grande Prairie, AB, Canada; Elias Lampakis, Kiparissia, Greece; Ronald Tiberio, Wellesley High School, Natick, MA; and Christian Yankov, Eastern Connecticut State University, Willimantic, CT, independently.

Since D and E are the respective points of contact of the incircle and excircle with the side BC of $\triangle ABC$, and I and X are the respective incenter and excenter, XE and ID are both perpendicular to BC. Because $|ID| = |GE|$, $|FD| = |XE|$, and