$$F \textit{ on } AB \textit{ and } G \textit{ on } CD \textit{ satisfy } \frac{AF}{FB} = \frac{DG}{GC}.$$

In fact, we shall see that if directed distances are used then F can be any point of the line AB and G the corresponding point on DC.

Let $T = EH \cap AB$; then $ET \perp AB$ (since ℓ is parallel to AB and perpendicular to EH). Because $ABCD$ is cyclic, the triangles DEC and AEB are oppositely similar. Because F and G are corresponding points in the similar triangles DEC and AEB as are K and T, we have $\angle EGK = \angle EFT$ or, using directed angles,

$$\angle KGE = \angle EFT. \tag{1}$$

We now set $S = EF \cap HK$ and want to prove that these lines are perpendicular at S. Because of the right angles at H and K, the points E, G, H, K lie on the circle whose diameter is EG, whence

$$\angle KHE = \angle KGE$$

(as directed angles). But $\angle SHE = \angle KHE$ (because $S \in KH$) and $\angle KGE = \angle EFT$ (from (1)), so it follows that $\angle SHE = \angle EFT$. Also, the vertically opposite angles at E are equal so that the triangles SHE and TFE are similar. But $\angle FTE = 90°$, hence $\angle HSE = 90°$; that is, $EF \perp HK$.

OC90. Let n be a positive integer. If one root of the quadratic equation $x^2 - ax + 2n = 0$ is equal to

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}},$$

prove that $2\sqrt{2n} \le a \le 3\sqrt{n}$.
Originally question 6 from the 2011 Kazakhstan National Olympiad, Grade 9.

Solved by G. Apostolopoulos; Š. Arslanagić; M. Bataille; C. Curtis; M. Dincă; O. Geupel; L. Giugiuc; B. Jin and E. T. H. Wang; N. Midttun; P. Perfetti; V. Pambuccian; G. Scărlătescu; D. Văcaru; and T. Zvonaru. We give a solution similar to those provided by most of the solvers.

Let

$$s := \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}.$$

Since

$$s^2 - as + 2n = 0$$

we have

$$a = s + \frac{2n}{s}.$$

Therefore, by the AM-GM inequality we get $a \ge 2\sqrt{2n}$.

To prove the other inequality we show first that $\sqrt{n} \le s \le 2\sqrt{n}$. The left hand side inequality is immediate:

$$s = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \ge \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}} = \sqrt{n}\,.$$

We next prove the right hand side inequality by induction. When $n = 1$, it states that $1 \le 2$. Next, assume the statement is true for some $n = k \ge 1$, so

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} \le 2\sqrt{k}.$$

Thus

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} &\le 2\sqrt{k} + \frac{1}{\sqrt{k+1}} = 2\sqrt{k} + \frac{2}{2\sqrt{k+1}} \\
&\le 2\sqrt{k} + \frac{2}{\sqrt{k} + \sqrt{k+1}} \\
&= 2\sqrt{k} + 2\left(\sqrt{k+1} - \sqrt{k}\right) = 2\sqrt{k+1}
\end{aligned}$$

which completes the induction.

Now, since $\sqrt{n} \le s \le 2\sqrt{n}$, we get

$$a = s + \frac{2n}{s} \le 2\sqrt{n} + \frac{n}{\sqrt{n}} = 3\sqrt{n}\,,$$

which completes the proof.

Editor's comment. As is often the case with a trigonometry problem, there were several different approaches exhibiting a variety of efficiency and recourse to other results. Kouba produced a solution similar to the second one above and Woo analyzed the graph of $y = 2\tan x - \sec x$ to show that it lay above the line $y = -\pi/3 + 2x$. Some solvers used the representation of the tangents and cosines of the half angles of a triangle in terms of the sides, semi-perimeter, inradius and area. Brown noted that the given inequality is equivalent to $s^2 \geq 12Rr + 3r^2$, while Lau reduced it to $3s^2 \leq (4R + r)^2$. Dincă proved this generalization : Let $A_1A_2\dots A_n$ be a convex $n-$gon. Then

$$\sum_{k=1}^{n} \tan\frac{A_k}{2} \geq \cos\frac{\pi}{n}\sum_{k=1}^{n}\sec\frac{A_k}{2}.$$

3777. [2012 : 335, 336] *Proposed by G. Apostolopoulos.*

Let x, y, and z be positive real numbers such that $xyz = 1$ and $\frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4} = 3$. Determine all possible values of $x^4 + y^4 + z^4$.

Solved by A. Alt ; Š.Arslanagic ; D. Bailey, E. Campbell and C. Diminnie ; M. Bataille ; C. Curtis ; R. Hess ; O. Kouba ; D. Koukakis ; S. Malikić (2 solutions) ; P. Perfetti ; A. Plaza ; C. M. Quang ; D. Smith ; D. R. Stone and J. Hawkins ; I. Uchiha ; D. Văcaru ; T. Zvonaru ; and the proposer. There was also an incorrect solution. We give a solution that is a composite of virtually all solutions received.

By the AM-GM Inequality, we have

$$3 = \frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4} \geq 3\sqrt[3]{\frac{1}{x^4}\cdot\frac{1}{y^4}\cdot\frac{1}{z^4}} = 3.$$

Thus, we must have the equality above, which implies that $\frac{1}{x^4} = \frac{1}{y^4} = \frac{1}{z^4}$ or $x = y = z$. Since we know that $xyz = 1$, it follows that $x = y = z = 1$ and so $x^4 + y^4 + z^4 = 3$.

3778. [2012 : 335, 337] *Proposed by M. Bataille.*

Let $\Delta A_1A_2A_3$ be a triangle with circumcentre O, incircle γ, incentre I, and inradius r. For $i = 1, 2, 3$, let A_i' on side A_iA_{i+1} and A_i'' on side A_iA_{i+2} be such that $A_i'A_i'' \perp OA_i$ and γ is the A_i–excircle of $\Delta A_iA_i'A_i''$ where $A_4 = A_1$, $A_5 = A_2$. Prove that

(a) $A_1'A_1'' \cdot A_2'A_2'' \cdot A_3'A_3'' = \dfrac{4a_1a_2a_3}{(a_1 + a_2 + a_3)^2} \cdot r^2$

(b) $A_1'A_1'' + A_2'A_2'' + A_3'A_3'' = \dfrac{a_1^2 + a_2^2 + a_3^2}{a_1a_2a_3} \cdot IK^2 + \dfrac{3a_1a_2a_3}{a_1^2 + a_2^2 + a_3^2}$

3780. [2012 : 335, 337] *Proposed by O. Furdui.*

Let $f : [0,1] \to \mathbb{R}$ be a continuously differentiable function and let

$$x_n = f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right).$$

Calculate $\lim\limits_{n\to\infty}(x_{n+1} - x_n)$.

Solved by A. Alt ; M. Bataille ; O. Kouba ; M. R. Modak ; P. Perfetti ; and the proposer. There were five flawed solutions, three of which applied an invalid converse of the Stolz-Cesaro theorem. We present 2 solutions.

Solution 1 by Omran Kouba.

The required limit is equal to $\int_0^1 f(x)dx$.

We first note that, if $g : [0,1] \longrightarrow \mathbf{R}$ is a continuously differentiable function, then, using integration by parts, we have that

$$\begin{aligned}\int_0^1 \left(x - \frac{1}{2}\right) g'(x)dx &= \left[\left(x - \frac{1}{2}\right) g(x)\right]_0^1 - \int_0^1 g(x)dx \\ &= \frac{g(1)+g(0)}{2} - \int_0^1 g(x)dx.\end{aligned}$$

Apply this to the function $g(x) = f((k+x)/n)$ for $k = 0, 1, 2, \ldots, n-1$ and add the resulting equations to obtain

$$x_n + \frac{f(0)+f(1)}{2} - n\int_0^1 f(x)dx = \int_0^1 \left(x - \frac{1}{2}\right) H_n(x)dx,$$

where

$$H_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f'\left(\frac{k+x}{n}\right).$$

Observe that, for each $x \in [0,1]$, $H_n(x)$ is a Riemann sum for the integral $\int_0^1 f'(t)dt$ and $|H_n(x)| \leq \sup_{[0.1]} |f'|$.

From the foregoing equation and its analogue for $n+1$, we obtain that

$$x_{n+1} - x_n - \int_0^1 f(x)dx = \int_0^1 (H_{n+1}(x) - H_n(x))\left(x - \frac{1}{2}\right)dx.$$

As n tends to infinity, the integrand on the right side tends pointwise and boundedly to 0, so by the Lebesgue Dominated Convergent Theorem, we conclude that $\lim_{n\to\infty}(x_{n+1} - x_n) = \int_0^1 f(x)dx$.

Solution 2 by Paolo Perfetti.

There exists $\xi_k \in (k(n+1)^{-1}, kn^{-1})$ for which

$$\begin{aligned} x_{n+1} - x_n &= \sum_{k=1}^{n} \left(f\left(\frac{k}{n+1}\right) - f\left(\frac{k}{n}\right) \right) + f(1) \\ &= \sum_{k=1}^{n} f'(\xi_k) \left(\frac{-k}{n(n+1)} \right) + f(1) \\ &= -\frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} \left(f'(\xi_k) - f'\left(\frac{k}{n}\right) \right) - \frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} f'\left(\frac{k}{n}\right) + f(1) \end{aligned}$$

where $k(n+1)^{-1} < \xi_k < kn^{-1}$. Note that f' is uniformly continuous on $[0,1]$ and that

$$\left| \xi_k - \frac{k}{n} \right| \le \frac{k}{n(n+1)} < \frac{1}{n}.$$

Therefore, for each $\epsilon > 0$, when n is sufficiently large

$$\left| f'(\xi_k) - f'\left(\frac{k}{n}\right) \right| < \epsilon$$

for $1 \le k \le n$. Thus

$$\left| -\frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} \left(f'(\xi_k) - f'\left(\frac{k}{n}\right) \right) \right| < \epsilon \left(\frac{1}{n(n+1)} \right) \left(\frac{n(n+1)}{2} \right) = \frac{\epsilon}{2}.$$

Moreover

$$\lim_{n\to\infty} -\frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} f'\left(\frac{k}{n}\right) = -\int_0^1 x f'(x) dx.$$

Therefore, integrating by parts, we find that

$$\lim_{n\to\infty} (x_{n+1} - x_n) = f(1) - \int_0^1 x f'(x) dx = \int_0^1 f(x) dx.$$

Editor's comment. Malikić and Ricardo noted that the proposer poses and solves this problem in his book *Limits, Series, and Fractional Part Integrals* published by Springer in 2013. It is problem 1.32 on page 6 under Miscellaneous Limits ; the solution appears on page 52.