Proposed solution of problem 3556 – deadline 03–01–2011

For any acute triangle with side lengths a, b and c, prove that

$$(a+b+c)\min\{a,b,c\} \le 2ab+2bc+2ca-a^2-b^2-c^2$$

Proof We introduce the well known change of variables $a = x+z$, $b = x+y$, $c = y+z$ or $x = (a+b-c)/2 > 0$, $y = (b+c-a)/2 > 0$, $z = (c+a-b)/2 > 0$. Moreover by the symmetry of the inequality we assume $a \le b \le c$ thus $a = \min\{a,b,c\}$ which means $y \ge z$, $y \ge x$. The inequality becomes

$$2(x+y+z)(x+z) \le 4(xy+yz+zx) \quad \Longleftrightarrow \quad (xy+zy) \ge x^2+z^2$$

which clearly holds because of the assumption on x, y, z.