Proposed solution of problem 3500

Define the function $f(a)=\sum_{k=1}^{\infty}\frac{\ln k}{k(k+a)}$ *for* $a\in(-1,+\infty)$ *and set* $\beta=-f(1)+\frac{1}{4}f(\frac{1}{2})-\frac{1}{4}f(-\frac{1}{2})$. *Prove that*

$$\prod_{k=1}^{\infty}\frac{(2k-1)^{\frac{1}{2k}}}{(2k)^{\frac{1}{2k-1}}}=2^{-\frac{3}{2}\ln(2)+1-\gamma}e^{\beta}$$

Proof We need the well known result

$$\sum_{k=1}^{n}\frac{1}{2k-1}=\frac{\ln}{2}+\ln 2+\frac{\gamma}{2}+o(1) \tag{1}$$

Doing the logarithm the product becomes

$$\ln\left[\prod_{k=1}^{\infty}\frac{(2k-1)^{\frac{1}{2k}}}{(2k)^{\frac{1}{2k-1}}}\right]=\sum_{k=1}^{n}\frac{\ln(2k-1)}{2k}-\sum_{k=1}^{n}\frac{\ln(2k)}{2k-1}$$

$$\sum_{k=1}^{n}\frac{\ln(2k-1)}{2k}=\sum_{k=1}^{2n}\frac{\ln k}{k+1}-\sum_{k=1}^{n}\frac{\ln(2k)}{2k+1}=\sum_{k=1}^{2n}\frac{\ln k}{k+1}-\sum_{k=1}^{n}\frac{\ln 2}{2k+1}-\sum_{k=1}^{n}\frac{\ln k}{2k+1} \tag{2}$$

$$-\sum_{k=1}^{n}\frac{\ln(2k)}{2k-1}=-\sum_{k=1}^{n}\frac{\ln 2}{2k-1}-\sum_{k=1}^{n}\frac{\ln k}{2k-1} \tag{3}$$

The sum of (2) and (3) yields

$$\sum_{k=1}^{2n}\frac{\ln k}{k+1}-\sum_{k=1}^{n}\frac{\ln k}{2k+1}-\sum_{k=1}^{n}\frac{\ln k}{2k-1}-\sum_{k=1}^{n}\frac{2\ln 2}{2k-1}+\ln 2-\frac{\ln 2}{n+1}$$

which we rewrite as

$$\sum_{k=1}^{n}\left(\frac{\ln k}{k+1}-\frac{\ln k}{k}\right)+\sum_{k=1}^{n}\left(\frac{\ln k}{2k}-\frac{\ln k}{2k+1}\right)+\sum_{k=1}^{n}\left(\frac{\ln k}{2k}-\frac{\ln k}{2k-1}\right)+ \\ +\sum_{k=n+1}^{2n}\frac{\ln k}{k+1}-\sum_{k=1}^{2n}\frac{2\ln 2}{2k-1}+\ln 2+o(1) \tag{4}$$

that is

$$-f(1)+\frac{1}{4}f(\frac{1}{2})-\frac{1}{4}f(-\frac{1}{2})+\sum_{k=n+1}^{2n}\frac{\ln k}{k+1}-\sum_{k=1}^{2n}\frac{2\ln 2}{2k-1}+\ln 2+o(1)= \\ \beta+\sum_{k=n+1}^{2n}\frac{\ln k}{k+1}-\sum_{k=1}^{2n}\frac{2\ln 2}{2k-1}+\ln 2+o(1) \tag{5}$$

The asymptotic behavior of monotonic series allows us to write

$$\sum_{k=n+1}^{2n} \frac{\ln k}{k+1} = \int_{n+1}^{2n} \frac{\ln x}{x+1} dx + o(1)$$

Now

$$\int_{n+1}^{2n} \frac{\ln x}{x+1} dx = \int_{n+1}^{2n} \ln x \left(\frac{1}{x+1} - \frac{1}{x} \right) dx + \int_{n+1}^{2n} \frac{\ln x}{x} dx$$

Since $\left| \ln x \left(\frac{1}{x+1} - \frac{1}{x} \right) \right| \leq C x^{-3/2}$ the first integral goes to zero for $n \to \infty$.

$$\int_{n+1}^{2n} \frac{\ln x}{x} dx = \frac{1}{2} \ln^2 x \Big|_{n+1}^{2n} = \frac{1}{2} \left(\ln^2(2n) - \ln^2(n+1) \right)$$
$$= \frac{1}{2} \big(\ln(2n) - \ln(n+1) \big) \big(\ln(2n) + \ln(n+1) \big) = \frac{1}{2} \ln 2 (\ln 2 + 2 \ln n) + o(1)$$

By inserting it into (5) and taking into account (1) we have

$$\beta + \frac{1}{2} \ln 2(\ln 2 + 2 \ln n) - 2 \ln 2 (\frac{1}{2} \ln n + \ln 2 + \frac{\gamma}{2}) + \ln 2 + o(1) =$$
$$= \beta - \frac{3}{2} \ln^2 2 - \gamma \ln 2 + \ln 2 = \beta + \ln 2 (-\frac{3}{2} \ln 2 - \gamma + 1)$$

Exponentiating we get

$$e^{\beta} \cdot 2^{-\frac{3}{2} \ln 2 - \gamma + 1}$$

concluding the proof.

Roma, (Italy) 03/19/2017

Thank you, Best regards
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