Proposed solution of problem 3495 – deadline 06–01–2010

Let a, b, c be real positive numbers with $a+b+c=2$. Prove that

$$\frac{1}{2}+\sum_{\text{cyclic}}\frac{a}{b+c}\leq\sum_{\text{cyclic}}\frac{a^2+bc}{b+c}\leq\frac{1}{2}+\sum_{\text{cyclic}}\frac{a^2}{b^2+c^2}$$

Proof L.H.S. Employing $a+b+c=2$ it is equivalent to

$$\left(\frac{1}{2}+\sum_{\text{cyclic}}\frac{a}{b+c}\right)\frac{a+b+c}{2}\leq\sum_{\text{cyclic}}\frac{a^2+bc}{b+c}$$

and clearing the denominators we obtain

$$\sum_{\text{sym}}(a^4+a^2bc)\geq\sum_{\text{sym}}(a^3b+a^2b^2)$$

Schür's inequality of second degree $a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)\geq 0$ yields

$$\sum_{\text{sym}}(a^4+a^2bc)\geq 2\sum_{\text{sym}}a^3b$$

(sym stands for symmetric) and then the inequality becomes

$$2\sum_{\text{sym}}a^3b\geq\sum_{\text{sym}}(a^3b+a^2b^2)$$

which holds being $\sum_{\text{sym}}a^3b\geq\sum_{\text{sym}}a^2b^2$ by the AGM's $(a^3b+b^3a)/2\geq a^2b^2$ and cyclic.

R.H.S. Homogenizing it is

$$\left(\frac{1}{2}+\sum_{\text{cyclic}}\frac{a^2}{b^2+c^2}\right)\frac{a+b+c}{2}\geq\sum_{\text{cyclic}}\frac{a^2+bc}{b+c}$$

Clearing the denominators the inequality is equivalent to

$$\sum_{\text{sym}}\left(2a^9b+4a^8bc+7a^7b^2c+a^7b^3+2a^4b^4c^2\right)\geq \\ \sum_{\text{sym}}\left(a^5b^5+2a^6b^4+5a^5b^3c^2+a^4b^3c^3+5a^5b^4c+2a^6b^3c\right) \tag{1}$$

Since $[9,1,0] \succ [6,4,0]$, $[7,3,0] \succ [5,5,0]$ and $[4,4,2] \succ [4,3,3]$ Muirhead's theorem guarantees that

$$\sum_{\text{sym}}(a^9b - a^6b^4) \geq 0, \quad \sum_{\text{sym}}(a^7b^3 - a^5b^5) \geq 0, \quad \sum_{\text{sym}}(a^4b^4c^2 - a^4b^3c^3) \geq 0$$

so that (1) is implied by

$$\sum_{\text{sym}}\left(4a^8bc + 7a^7b^2c + a^4b^4c^2\right) \geq \sum_{\text{sym}}\left(5a^5b^3c^2 + 5a^5b^4c + 2a^6b^3c\right) \tag{2}$$

Now by the AGM we have $(a^7b^2c + a^4b^4c^2) \geq 2a^{\frac{11}{2}}b^3c^{\frac{3}{2}}$ thus

$$\sum_{\text{sym}}(4a^8bc + 7a^7b^2c + a^4b^4c^2) \geq \sum_{\text{sym}}(4a^8bc + 6a^7b^2c + 2a^{\frac{11}{2}}b^3c^{\frac{3}{2}})$$

so that (2) is implied by

$$\sum_{\text{sym}}(4a^8bc + 6a^7b^2c + 2a^{\frac{11}{2}}b^3c^{\frac{3}{2}}) \geq \sum_{\text{sym}}(5a^5b^3c^2 + 5a^5b^4c + 2a^6b^3c)$$

A multiple application of Muirhead's inequality completes the proof since: $[\frac{11}{2}, 3, \frac{3}{2}] \succ [5,3,2]$, $[7,2,1] \succ [6,3,1]$, $[7,2,1] \succ [5,4,1]$, $[8,1,1] \succ [6,3,1]$, $[8,1,1] \succ [5,3,2]$

To complete the proof we give the AGM's underlying the applications of Muirhead's theorem (for each application we write only one of the six inequalities relative to the complete proof)

$[9,1,0] \succ [6,4,0]$: $(5a^9b + 3b^9a)/8 \geq a^6b^4$,

$[7,3,0] \succ [5,5,0]$: $(a^7b^3 + a^3b^7)/2 \geq a^5b^5$

$[4,4,2] \succ [4,3,3]$: $(2a^4b^4c^2 + 2a^4b^2c^4)/4 \geq a^4b^3c^3$,

$[\frac{11}{2}, 3, \frac{3}{2}] \succ [5,3,2]$: $(41a^{\frac{11}{2}}b^3c^{\frac{3}{2}} + 5c^{\frac{11}{2}}a^3b^{\frac{3}{2}} + 3b^{\frac{11}{2}}c^3a^{\frac{3}{2}})/49 \geq a^5b^3c^2$

$[7,2,1] \succ [6,3,1]$: $(4a^7b^2c + a^2b^7c)/5 \geq a^6b^3c$

$[7,2,1] \succ [5,4,1]$: $(3a^7b^2c + 2a^2b^7c)/5 \geq a^5b^4c$

$[8,1,1] \succ [6,3,1]$: $(5a^8bc + 2ab^8c)/7 \geq a^6b^3c$

$[8,1,1] \succ [5,3,2]$: $(4a^8bc + 2ab^8c + c^8ab)/7 \geq a^5b^3c^2$