Proposed solution of problem 3488 – deadline 06–01–2010

Let a, b and c be positive real numbers. Prove that

$$\frac{a}{2a^2+bc}+\frac{b}{2b^2+ac}+\frac{c}{2c^2+ab}\le\sqrt{\frac{a^{-1}+b^{-1}+c^{-1}}{a+b+c}}$$

Proof.

By squaring, the inequality is equivalent to

$$\sum_{\text{sym}} 23a^7b^4c^3+16a^6b^6c^2+12a^8b^4c^2+12a^8b^5c+4a^7b^6c+4a^9b^3c^2+8a^7b^7\ge$$
$$\sum_{\text{sym}}\frac{33}{2}a^5b^5c^4+22a^6b^5c^3+26a^6b^4c^4+12a^7b^5c^2+\frac{5}{2}a^8b^3c^3$$

Then the solution comes from a multiple application of Muirhead's theorem. We show the majorizations and the underlying AGM's.

$$[8,4,2]\succ[7,5,2],\quad (3a^8b^4c^2+a^4b^8c^2)/4\ge a^7b^5c^2$$

and the inequality becomes

$$\sum_{\text{sym}} 23a^7b^4c^3+16a^6b^6c^2+12a^8b^4c^2+4a^7b^6c+4a^9b^3c^2+8a^7b^7\ge$$
$$\sum_{\text{sym}}\frac{33}{2}a^5b^5c^4+22a^6b^5c^3+26a^6b^4c^4+\frac{5}{2}a^8b^3c^3$$

$$[7,4,3,]\succ[6,5,3]\quad (2a^7b^4c^3+a^4b^7c^3)/3\ge a^6b^5c^3$$
$$[7,4,3,]\succ[6,4,4]\quad (3a^7b^4c^3+a^3b^4c^7)/4\ge a^6b^4c^4$$

and then we get

$$\sum_{\text{sym}} 16a^6b^6c^2+12a^8b^4c^2+4a^7b^6c+4a^9b^3c^2+8a^7b^7\ge\sum_{\text{sym}}\frac{33}{2}a^5b^5c^4+25a^6b^4c^4+\frac{5}{2}a^8b^3c^3$$

$$[8,4,2,]\succ[8,3,3]\quad (a^8b^4c^2+a^8b^2c^4)/2\ge a^8b^3c^3$$
$$[8,4,2,]\succ[6,4,4]\quad (2a^8b^4c^2+a^2b^4c^8)/4\ge a^6b^4c^4$$

and then we get

$$\sum_{\text{sym}} 16a^6b^6c^2+4a^7b^6c+4a^9b^3c^2+8a^7b^7\ge\sum_{\text{sym}}\frac{33}{2}a^5b^5c^4+\frac{31}{2}a^6b^4c^4$$

$$[6,6,2] \succ [6,4,4] \quad (a^6b^6c^2 + a^6b^2c^6)/2 \geq a^6b^4c^4$$

and then we get

$$\sum_{\text{sym}} \frac{1}{2}a^6b^6c^2 + 4a^7b^6c + 4a^9b^3c^2 + 8a^7b^7 \geq \sum_{\text{sym}} \frac{33}{2}a^5b^5c^4$$

$$[6,6,2] \succ [5,5,4], \quad (abc)^2(2a^4b^4 + a^4c^4 + b^4c^4)/4 \geq (a^3b^3c^2)(abc)^2$$
$$[7,6,1] \succ [5,5,4] \quad (abc)(14a^6b^5 + 9b^6c^5 + 8c^6a^5)/31 \geq a^4b^4c^3(abc)$$
$$[9,3,2] \succ [5,5,4] \quad (abc)(17a^8b^2c + 10a^2bc^8 + 16ab^8c^2)/43 \geq a^4b^4c^3(abc)$$
$$[7,7,0] \succ [5,5,4] \quad (3a^7b^7 + 2b^7c^7 + 2a^7c^7)/7 \geq a^5b^5c^4$$

concluding the proof.