Proposed solution of problem 3486 – deadline 05–01–2010

Let a, b, c be positive real numbers. Prove that

$$\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab}\le\frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

Proof The inequality is

$$\sum_{\text{cyc}}\left(1-\frac{a^2}{a^2+bc}\right)\le\frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

or

$$\frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}+\sum_{\text{cyc}}\frac{a^2}{a^2+bc}\ge 3$$

Cauchy–Schwarz yields

$$\sum_{\text{cyc}}\frac{a^2}{a^2+bc}\ge\frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca}=\frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca}$$

and then

$$\frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}+\frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)}\ge 3$$

Now set $a+b+c=1$ (the inequality is homogeneous) and define $ab+bc+ca=\dfrac{1-x^2}{3}$, $0\le x\le 1$. We know that (Matematical Reflections, vol.2007, issue 2, "On a class of three–variable inequalities", Vo Quoc Ba Can), we are interested in the r.h.s.

$$\frac{(1+x)^2(1-2x)}{27}\le abc\le\frac{(1-x)^2(1+2x)}{27}$$

In the variable x the inequality reads as

$$\frac{1}{2+x^2}+\frac{1}{2}\sqrt[3]{\frac{1+x}{(1-x)(1+2x)}}\ge 1$$

or

$$\frac{1}{8}\frac{1+x}{(1-x)(1+2x)}-\left(1-\frac{1}{2+x^2}\right)^3\ge 0$$

and after simple algebra we come to

$$-x^2(16x^6-7x^5+41x^4-18x^3+30x^2+4)\le 0$$

which evidently holds taking into account that $0\le x\le 1$ and then $30x^2\ge 18x^3$, $41x^4\ge 7x^5$.