Proposed solution of problem 3388 – deadline 05–01–09

For all $x \geq 1$ *show that*

$$\frac{1}{2}\sqrt{x-1} + \frac{(x-1)^2}{\sqrt{x-1}+\sqrt{x+1}} < \frac{x^2}{\sqrt{x}+\sqrt{x+2}}$$

Proof Since $\sqrt{x+1} > \sqrt{x-1}$ and $\sqrt{x}+\sqrt{x+2} < \sqrt{2}\sqrt{2x+2}$ by the concavity of $\sqrt{x}$, we get

$$\frac{1}{2}\sqrt{x-1} + \frac{1}{2}(x-1)^{3/2} < \frac{x^2}{2\sqrt{x+1}}$$

or

$$\sqrt{x^2-1} + \sqrt{x^2-1}(x-1) < x^2$$

By $ab \leq (a^2+b^2)/2$ we have $\sqrt{x^2-1}(x-1) < (x^2-1+(x-1)^2)/2$ yielding

$$\sqrt{x^2-1} + \frac{1}{2}(2x^2-2x) < x^2 \qquad \text{or} \qquad \sqrt{x^2-1} < x.$$