Proposed solution of problem 3387 – deadline 05–01–09

Dear Editor of Crux Mathematicorum with Mathematical Myhem, I would like to submit the following solution of problem 3387 (deadline 05–01–09)

Let $k > l \geq 0$ be fixed integers. Find

$$\lim_{x\to\infty} 2^x \left(\zeta(x+k)^{\zeta(x+k)} - \zeta(x+l)^{\zeta(x+l)}\right)$$

where ζ is the Riemann Zeta function

Answer: $2^{-k} - 2^{-l}$

Proof $\quad \zeta(n) = 1 + 2^{-n} + 3^{-n} + \sum_{k=4}^{\infty} k^{-n}$ and

$$\frac{1}{(n-1)4^{n-1}} = \int_4^{+\infty} \frac{dy}{y^n} \leq \sum_{k=4}^{\infty} k^{-n} \leq \int_3^{+\infty} \frac{dy}{y^n} = \frac{1}{(n-1)3^{n-1}}$$

hence $\sum_{k=4}^{\infty} k^{-n} = O(3^{-n})$ and $\zeta(n) = 1 + 2^{-n} + O(3^{-n})$. Moreover

$$\begin{aligned}\zeta(n)^{\zeta(n)} &= \exp\Big((1+2^{-n}+O(3^{-n}))\cdot\ln\big(1+2^{-n}+O(3^{-n})\big)\Big) = \\ &= \exp\Big((1+2^{-n}+O(3^{-n}))\cdot(2^{-n}+O(3^{-n}))\Big) = \exp(2^{-n}+O(3^{-n})) = \\ &= 1+2^{-n}+O(3^{-n}))\end{aligned}$$

Finally we have

$$\begin{aligned}&\lim_{x\to\infty} 2^x \left(\zeta(x+k)^{\zeta(x+k)} - \zeta(x+l)^{\zeta(x+l)}\right) = \lim_{x\to\infty} 2^x \left(2^{-x-k} - 2^{-x-l} + O(3^{-x-l})\right) = \\ &= 2^{-k} - 2^{-l}\end{aligned}$$