Proposed solution of problem 3370 – deadline 04–01–09

Let a_i and b_i be positive real numbers for $1 \le i \le k$, and let n be a positive integer. Prove that

$$\left(\sum_{j=1}^{k} a_i^{\frac{1}{n}}\right)^n \le \left(\sum_{j=1}^{k} \frac{a_i}{b_i}\right)\left(\sum_{j=1}^{k} a_i^{\frac{1}{n-1}}\right)^{n-1}$$

Proof We employ Holder's inequality

$$\left|\sum_{i=1}^{k} \alpha_i \beta_i\right| \le \left|\sum_{i=1}^{k} \alpha_i^n\right|^{\frac{1}{n}} \left|\sum_{i=1}^{k} \beta_i^{\frac{n}{n-1}}\right|^{\frac{n-1}{n}}$$

hence

$$\left(\sum_{j=1}^{k} \left(\frac{a_i}{b_i}\right)^{\frac{1}{n}} b_i^{\frac{1}{n}}\right)^n \le \left(\sum_{j=1}^{k} \left(\frac{a_i}{b_i}\right)^{n\frac{1}{n}}\right)^{n\frac{1}{n}} \left(\sum_{j=1}^{k} \left(b_i^{\frac{1}{n}\frac{n}{n-1}}\right)\right)^{n\frac{n-1}{n}} =$$

$$\left(\sum_{j=1}^{k} \frac{a_i}{b_i}\right)\left(\sum_{j=1}^{k} a_i^{\frac{1}{n-1}}\right)^{n-1}$$

For $n = 1$ trivially we have $\sum_{i=1}^{k} \frac{a_i}{b_i} b_i \le \sum_{i=1}^{k} \frac{a_i}{b_i} \sum_{i=1}^{k} b_i$ but the inequality is undefined. The proof is complete.