Proposed solution of prob. 1256

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Dear Professor, I would like to submit the included solution of problem 1256 ΠME Journal (Spring 2012), (deadline 10–01–2012):

Let

$$h(i)=\sum_{j=0}^{i}\frac{1}{2j+1}=1+\frac{1}{3}+\dots\frac{1}{2i+1}$$

Show that, for all nonnegative integers k

$$\sum_{i=0}^{k}\frac{1}{2(k-i)+1}h(i)=2\sum_{i=0}^{k}\frac{1}{2i+2}h(i)$$

Proof Induction. For $k=0$ trivially holds. Let's suppose it true for $0\leq k\leq n$ namely

$$\sum_{i=0}^{k}\frac{1}{2(k-i)+1}h(i)=\sum_{i=0}^{k}\frac{1}{i+1}h(i)$$

For $k=n+1$ we need to prove

$$\sum_{i=0}^{n+1}\frac{1}{2(n+1-i)+1}h(i)=\sum_{i=0}^{n+1}\frac{1}{i+1}h(i)$$

By changing $n+1-i=r$ we get

$$\sum_{i=0}^{n+1}\frac{1}{2(n+1-i)+1}h(i)=\sum_{r=0}^{n+1}\frac{1}{2r+1}h(n+1-r)$$

and by definition of $h(i)$

$$\sum_{r=0}^{n+1}\frac{1}{2r+1}h(n+1-r)=\sum_{r=0}^{n}\frac{1}{2r+1}h(n-r)+\sum_{r=0}^{n}\frac{1}{2r+1}\frac{1}{2n-2r+3}+\frac{1}{2n+3}h(0)$$

We need to show that

$$\sum_{r=0}^{n} \frac{1}{2r+1} h(n-r) + \sum_{r=0}^{n} \frac{1}{2r+1} \frac{1}{2n-2r+3} + \frac{1}{2n+3} =$$
$$= \sum_{i=0}^{n} \frac{1}{i+1} h(i) + \frac{h(n+1)}{n+2}$$

which by virtue of the induction hypotheses becomes

$$\sum_{r=0}^{n} \frac{1}{2r+1} \frac{1}{2n-2r+3} + \frac{1}{2n+3} = \frac{h(n+1)}{n+2}$$

or

$$\frac{1}{2(n+2)} \sum_{r=0}^{n} \frac{1}{2r+1} + \frac{1}{2(n+2)} \sum_{r=0}^{n} \frac{1}{2n-2r+3} + \frac{1}{2n+3} = \frac{h(n+1)}{n+2}$$

We can rewrite it as

$$\frac{1}{2(n+2)} \left(h(n) + h(n) + \frac{1}{2n+3} - 1 \right) + \frac{1}{2n+3} = \frac{h(n+1)}{n+2}$$

or

$$\frac{1}{2(n+2)} \left(2h(n) + \frac{1}{2n+3} - 1 \right) + \frac{1}{2n+3} = \frac{h(n)}{n+2} + \frac{1}{(2n+3)(n+2)}$$

or

$$\frac{1}{2(n+2)} \left(\frac{1}{2n+3} - 1 \right) + \frac{1}{2n+3} = \frac{1}{(2n+3)(n+2)}$$

which evidently holds true and this concludes the proof.

Roma 04/27/2012

Best regards
Paolo Perfetti